Fall 2019: Classical Field Theory (PH6297)

$U(1)$ Symmetry of the complex scalar and scalar electrodynamics

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1 Global $U(1)$ symmetry of the complex field theory & associated Noether charge

Consider the complex scalar field theory, defined by the action,

$$I\left[\Phi(x),\Phi^\dagger(x)\right]=\int d^4x\,\left[(\partial_\mu\Phi)^\dagger\,\partial^\mu\Phi-V\left(\Phi^\dagger\Phi\right)\right].\tag{1}$$

As we have noted earlier complex scalar field theory action Eq. (1) is invariant under multiplication by a **constant** complex phase factor $e^{i\,\alpha}$,

$$\begin{aligned}\Phi\to\Phi' &= e^{-i\,\alpha}\Phi,\\ \Phi^\dagger\to\Phi'^\dagger &= e^{i\,\alpha}\Phi^\dagger.\end{aligned}\tag{2}$$

The phase,α is necessarily a real number. Since a complex phase is unitary 1×1 matrix i.e. the complex conjugation is also the inverse,

$$\left(e^{-i\,\alpha}\right)^\dagger=\left(e^{-i\,\alpha}\right)^{-1},$$

such phases are also called $U(1)$ factors (U stands for Unitary matrix and since a number is a 1×1 matrix, $U(1)$ is unitary matrix of size 1×1). Since this symmetry transformation does not touch spacetime but only changes the fields, such a symmetry is called an **internal symmetry**. Also note that since α is a constant i.e. not a function of spacetime, it is a **global** symmetry (**global = same everywhere = independent of spacetime location**).

Check: Under the U(1) symmetry Eq. (2), the combination $\Phi^\dagger\Phi$ is obviously invariant,

$$\begin{aligned}\Phi'^\dagger\Phi' &= \left(e^{i\,\alpha}\Phi^\dagger\right)\left(e^{-i\,\alpha}\Phi\right)\\ &= \Phi^\dagger\Phi.\end{aligned}$$

This implies any function of the product $\Phi^\dagger\Phi$ is also invariant.

$$V\left(\Phi'^\dagger\Phi'\right)=V\left(\Phi^\dagger\Phi\right).$$

Note that this is true whether α is a constant or a function of spacetime i.e. $\alpha(x)$.

Next let's look at the kinetic term,

$$\begin{aligned}\left(\partial_\mu \Phi^\dagger\right)(\partial^\mu \Phi) \rightarrow \left(\partial_\mu \Phi'^\dagger\right)(\partial^\mu \Phi') &= \partial_\mu \left(e^{i\,\alpha}\Phi^\dagger\right)\partial^\mu\left(e^{-i\,\alpha}\Phi\right), \\ &= e^{i\,\alpha}\left(\partial_\mu \Phi^\dagger\right)\, e^{-i\,\alpha}\left(\partial^\mu \Phi\right) \\ &= \left(\partial_\mu \Phi^\dagger\right)(\partial^\mu \Phi).\end{aligned}$$

So this kinetic term in the action is also invariant because α is a constant and the derivative does not act on it. If α was a function of spacetime, $\alpha = \alpha(x)$, the derivative would have acted on it and the term would not be invariant. Incidentally, a spacetime dependent phase $\alpha(x)$ is called a **local $U(1)$ transformation.**

1.1 Conserved charges corresponding to the $U(1)$ symmetry

Here we obtain the conserved charge for the global $U(1)$ symmetry of the complex scalar field using the Noether method. As a first step in the process, one expresses the $U(1)$ symmetry transformation of the field (and its complex conjugate) in infinitesimal form,

$$\begin{aligned}\Phi \rightarrow \Phi' &= e^{-i\,\alpha}\Phi \\ &= (1 - i\alpha + O(\alpha^2))\Phi \\ &\approx \Phi - i\alpha\Phi,\end{aligned}$$

while the complex conjugate field (to first order in α) changes to,

$$\Phi'^\dagger \approx \Phi^\dagger + i\alpha\Phi^\dagger.$$

Next step in the process is to ***temporarily assume***, α is a function of spacetime,

$$\alpha = \alpha(x).$$

But since $\alpha = \alpha(x)$ is not a symmetry of the action, the action will change if we replace, $\Phi \rightarrow \Phi' = \Phi - i\alpha(x)\,\Phi$, in the action (1) i.e. $I\left[\Phi', \Phi'^\dagger\right] \neq I\left[\Phi, \Phi^\dagger\right]$. The next step in the Noether method is to compute the change in the action, $\delta I = I\left[\Phi', \Phi'^\dagger\right] - I\left[\Phi, \Phi^\dagger\right]$. For that we first need to find the change in the derivative of the field,

$$\begin{aligned}\partial^\mu \Phi \rightarrow \partial^\mu \Phi' &= \partial^\mu\left(\Phi - i\alpha(x)\,\Phi\right) \\ &= \partial^\mu \Phi - i\left(\partial^\mu \alpha\right)\Phi - i\alpha(x)\,\partial^\mu \Phi,\end{aligned}$$

and the derivative of the complex conjugate,

$$\begin{aligned}\partial_\mu \Phi^\dagger \rightarrow \partial_\mu \Phi'^\dagger &= \partial_\mu\left(\Phi^\dagger + i\alpha(x)\Phi^\dagger\right) \\ &= \partial_\mu \Phi^\dagger + i\left(\partial_\mu \alpha\right)\Phi^\dagger + i\alpha(x)\,\partial_\mu \Phi^\dagger.\end{aligned}$$

Using these expressions, the action for the transformed fields is,

$$
\begin{aligned}
I\left[\Phi', \Phi'^{\dagger}\right] &= \int d^4x \, \left[\left(\partial_\mu \Phi'\right)^{\dagger} \partial^\mu \Phi' - V(\Phi'^{\dagger}\Phi')\right] \\
&= \int d^4x \, \left[\left(\partial_\mu \Phi^{\dagger} + i\left(\partial_\mu \alpha\right)\Phi^{\dagger} + i\alpha(x)\,\partial_\mu \Phi^{\dagger}\right)\left(\partial^\mu \Phi - i\left(\partial^\mu \alpha\right)\Phi - i\alpha(x)\,\partial^\mu \Phi\right) - V(\Phi^{\dagger}\Phi)\right] \\
&= \int d^4x \, \left[\left(\partial_\mu \Phi\right)^{\dagger} \partial^\mu \Phi - V(\Phi^{\dagger}\Phi) - i\partial_\mu \alpha\left(\Phi \partial_\mu \Phi^{\dagger} - \Phi^{\dagger}\partial^\mu \Phi\right)\right] \\
&= I\left[\Phi, \Phi^{\dagger}\right] - \int d^4x \, \partial_\mu \alpha \, i\left(\Phi \partial_\mu \Phi^{\dagger} - \Phi^{\dagger}\partial^\mu \Phi\right).
\end{aligned}
$$

So the first order (in α) change in the action is,

$$
\begin{aligned}
\delta I &= I\left[\Phi', \Phi'^{\dagger}\right] - I\left[\Phi, \Phi^{\dagger}\right] \\
&= -\int d^4x \, \partial_\mu \alpha(x) \, i\left(\Phi \partial_\mu \Phi^{\dagger} - \Phi^{\dagger}\partial^\mu \Phi\right).
\end{aligned}
$$

From this expression we can identify the conserved current corresponding to the global U(1) symmetry,

$$
j^\mu = i\left(\Phi \partial_\mu \Phi^{\dagger} - \Phi^{\dagger}\partial^\mu \Phi\right). \tag{3}
$$

One can easily check that is conserved on-shell (on-shell means the classical equation of motion holds). For example, for the free theory, $V(\Phi^{\dagger}\Phi) = m^2\Phi^{\dagger}\Phi$, and we have,

$$
\begin{aligned}
\partial_\mu j^\mu &= i\left(\Phi^{\dagger}\partial^2\Phi - \Phi\partial^2\Phi^{\dagger}\right) \\
&= i\left(-\Phi^{\dagger}m^2\Phi + \Phi m^2\Phi^{\dagger}\right) \\
&= 0.
\end{aligned}
$$

Here we have used the equation of motion for the free complex scalar field (the Klein-Gordon equation)

$$
\left(\partial^2 + m^2\right)\Phi = \left(\partial^2 + m^2\right)\Phi^{\dagger} = 0. \tag{4}
$$

Finally, the conserved charge is then given by the volume integral,

$$
Q = \int d^3\mathbf{x} \, j^0 = i\int d^3\mathbf{x} \, \left(\Phi\dot{\Phi}^{\dagger} - \Phi^{\dagger}\dot{\Phi}\right). \tag{5}
$$

Homework: Check that the current (3) is conserved i.e. obeys the continuity equation $\partial_\mu j^\mu = 0$ not just for the free case i.e. when $V = m^2\Phi^{\dagger}\Phi$ but for a potential which is a more general function of $\Phi^{\dagger}\Phi$, i.e. $V(\Phi^{\dagger}\Phi)$. (Hint: Use the Euler-Lagrange equation of motion).

1.1.1 The charge conjugation symmetry of the complex scalar field theory

Note that in addition to the continuous global $U(1)$ symmetry (2), there is another discrete internal symmetry of the complex field theory, namely, interchanging the field Φ with its complex conjugate, $\Phi^{\dagger}$,

$$
\Phi \leftrightarrow \Phi^{\dagger}.
$$

Under this symmetry of course the charge (polarity) of the scalar field also changes,

$$
Q \to -Q.
$$

This is why this discrete symmetry is dubbed as ***charge conjugation symmetry***. In the quantum theory, this discrete internal symmetry will transform particles to antiparticles and vice-versa.

2 Coupling complex scalar to the Maxwell field: Scalar Electrodynamics

Here we try to couple the complex scalar field to the Maxwell field with the anticipation that the global $U(1)$ charge of the scalar is in fact the electric charge. For simplicity we will work with the free scalar i.e. take the complex scalar lagrangian to be,

$$\mathcal{L}_\Phi = \left(\partial^\mu \Phi^\dagger\right)(\partial_\mu \Phi) - m^2 \Phi^\dagger \Phi, \tag{6}$$

which leads to the equations of motion (4). The $U(1)$ symmetry of this lagrangian is,

$$\Phi \to \Phi' = e^{-ig\alpha}\Phi,$$

where we have introduced g as a quantum of charge (even before quantizing the theory) and α is the $U(1)$ parameter. This leads to the Noether current, $g\, j^\mu_{(0)}$, where

$$j^\mu_{(0)} = i\left(\Phi \partial_\mu \Phi^\dagger - \Phi^\dagger \partial^\mu \Phi\right). \tag{7}$$

The significance of the subscript 0 in the expression for current will become clear in what follows (as the leading term in a perturbation expansion in powers of the "coupling constant", g, with which the scalar matter couples to the Maxwell field).

The free Maxwell field on the other hand is described by the lagrangian,

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{8}$$

This has a *gauge* symmetry, namely,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \lambda(x). \tag{9}$$

Thus the full lagrangian describing the dynamics of scalar fields interacting with the Maxwell field (scalar electrodynamics) must contain these two terms at least, plus an interaction term, call it $\mathcal{L}'$

$$\mathcal{L}_{SED} = \mathcal{L}_{(0)} + \mathcal{L}', \quad \mathcal{L}_{(0)} = \mathcal{L}_\Phi + \mathcal{L}_A \tag{10}$$

Since we want to identify the Noether current for global $U(1)$ symmetry of the scalar field with the electric current that couples to Maxwell field, we are lead to write down an interaction term,

$$\mathcal{L}' = -g j^\mu_{(0)} A_\mu.$$

Here g is a dimensionless coupling constant which governs how strongly the scalar field interacts with the Maxwell gauge field. Thus the full interacting lagrangian for scalar electrodynamics could be,

$$\mathcal{L}_{SED} = \mathcal{L}_\Phi + \mathcal{L}_A + \mathcal{L}'.$$

Now let's look at the equations of motion of this system. The equation of motion for the Maxwell field is given by,

$$\begin{aligned}
\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)}\right) &= \frac{\partial \mathcal{L}}{\partial A_\nu} \\
\implies \partial_\mu \left(\frac{\partial \mathcal{L}_A}{\partial\left(\partial_\mu A_\nu\right)}\right) &= \frac{\partial \mathcal{L}'}{\partial A_\nu} \\
\implies \partial_\mu F^{\mu\nu} &= g j^\nu_{(0)}.
\end{aligned} \tag{11}$$

Taking the partial derivative ∂_ν of both sides of the above equation,

$$\partial_\mu \partial_\nu F^{\mu\nu} = g \partial_\mu j^\mu_{(0)}.$$

The LHS vanishes on account of symmetry (the derivatives $\partial_\mu \partial_\nu$ are symmetric under exchange $\mu \leftrightarrow \nu$, while the field strength $F^{\mu\nu}$ is antisymmetric under the same exchange $\mu \leftrightarrow \nu$), and so must the RHS, i.e. one must have $\partial_\mu j^\mu_{(0)} = 0$. Thus the Maxwell equation (11) is consistent only when the current on the RHS is a conserved one.

The scalar equation of motion now is given by,

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Phi^\dagger \right)} \right] = \frac{\partial \mathcal{L}}{\partial \Phi^\dagger},$$

or, since

$$\partial_\mu \left[\frac{\partial \mathcal{L}_\Phi}{\partial \left(\partial_\mu \Phi^\dagger \right)} \right] + \partial_\mu \left[\frac{\partial \mathcal{L}'}{\partial \left(\partial_\mu \Phi^\dagger \right)} \right] = - \frac{\partial (m^2 \Phi^\dagger \Phi)}{\partial \Phi^\dagger} + \frac{\partial \mathcal{L}'}{\partial \Phi^\dagger}$$

which gives,

$$\Box \Phi - ig \partial_\mu \left(\Phi A^\mu \right) = -m^2 \Phi + ig \left(\partial_\mu \Phi \right) A^\mu$$

or,

$$\left(\Box + m^2 \right) \Phi = 2ig\, A^\mu \, \partial_\mu \Phi + ig\, \Phi \, \partial_\mu A^\mu \tag{12}$$

Similarly for the complex conjugate field $\Phi^\dagger$,

$$\left(\Box + m^2 \right) \Phi^\dagger = -2ig A_\mu \partial^\mu \Phi^\dagger - ig \Phi^\dagger \partial^\mu A_\mu$$

So the scalar equation of motion has changed! Now recall that the conservation of the $U(1)$ current $j^\mu_{(0)}$ defined in (7) depended on the equation of motion of the scalar field be the Klein-Gordon equation and not the new equation (12). To check if the current is conserved with the new scalar equation (12), we take a 4-divergence of the current (7),

$$\begin{aligned} \partial_\mu j^\mu_{(0)} &= ig \left(\Phi \Box \Phi^\dagger - \Phi^\dagger \Box \Phi \right) \\ &= 2g^2 \partial_\mu \left(A^\mu \Phi^\dagger \Phi \right) \\ &\neq 0. \end{aligned}$$

So the current $j^\mu_{(0)}$ is not conserved, and hence the Maxwell equation (11) is inconsistent with the scalar equation (12). To make things consistent one needs to put on the RHS of the Maxwell equation (11) a ***strictly conserved current.***

The reason for failure of conservation of the current, $j_{(0)}$ is simply because it is not the Noether current of the full interacting Lagrangian (10), instead just of the pure scalar part, $\mathcal{L}_\Phi$. Thus the equations of motion which follow from the trial lagrangian, (10), (in particular the interaction term, $\mathcal{L}'$) are inconsistent, especially the Maxwell equation (11) must contain on the RHS the *full* Noether current. To get to the correct interaction lagrangian we will follow the Noether procedure in which we temporarily turn the global $U(1)$ parameter, α in a local one i.e. an arbitrary function of spacetime, $\alpha(x)$. In the lowest order (in powers of g), the transformation of the complex scalar is then,

$$\delta \Phi(x) = \Phi'(x) - \Phi(x) = -i\, g\, \alpha(x) \Phi(x)$$

Our starting point is the lagrangian,

$$\mathcal{L}_{SED} = \mathcal{L}_{\Phi} + \mathcal{L}_A + \mathcal{L}'$$

where the consistent interaction term, $\mathcal{L}'$ is now yet to be *determined*. Through the Noether procedure we will determine$\mathcal{L}'$ as a power series (i.e. perturbatively) in g, namely,

$$\mathcal{L}' = \sum_{n=0}^{\infty} g^n \, \mathcal{L}'_{(n)}.$$

The steps of the procedure are as follows:

1. Identify the gauge transformation parameter of the gauge field, λ with the Noether symmetry parameter, α for the matter field.

   $$\lambda = C\,\alpha$$

   where C is an undetermined constant. So now we are considering the joint/simultaneous transformation,

   $$\begin{aligned} \delta\Phi &= e^{-i\,g\,\alpha(x)}\Phi(x), \\ \delta A_\mu &= -C\,\partial_\mu \alpha(x). \end{aligned} \tag{13}$$

2. Find out the variation in the *non-interacting Lagrangian* under the joint transformation (13),

   $$\delta\mathcal{L}_{(0)} = \delta\mathcal{L}_{\Phi} + \delta\mathcal{L}_A = \delta\mathcal{L}_{\Phi} = -g\partial_\mu \alpha \, j^\mu_{(0)}.$$

3. Now add a term to the non-interaction lagrangian $\mathcal{L}_{(0)}$ which is order 1 in g i.e. linear in g such that the variation of it should cancel the variation of the zeroth order i.e.

   $$\begin{aligned} g\,\delta\mathcal{L}_{(1)} &= -\delta\mathcal{L}_{(0)} \\ &= g\,\partial_\mu \alpha\, j^\mu_{(0)} \\ &= -\frac{g}{C}\delta A_\mu \, j^\mu_{(0)}. \end{aligned}$$

   If we omit terms which are $O(g^2)$, then,

   $$\delta\left(-\frac{g}{C}A_\mu \, j^\mu_{(0)}\right) = g\delta\mathcal{L}_{(1)} + O(g^2)$$

   because,

   $$\delta j^\mu_{(0)} = -2g\,\left(\Phi^\dagger\Phi\right)\,\partial^\mu\alpha \sim O(g).$$

   So we can declare,

   $$\mathcal{L}_{(1)} = -\frac{1}{C}A_\mu j^\mu_{(0)}.$$

   Thus to linear order in g, the interacting Lagrangian is,

   $$\mathcal{L} = \mathcal{L}_{(0)} + g\,\mathcal{L}_{(1)}$$

   Now if we choose $C = 1$, then this interaction term gives the correct potential energy in the electrostatics limit. So we will choose $C = 1$ from now on.

4. Now the full lagrangian, $\mathcal{L}_{(0)} + \mathcal{L}_{(1)}$ is invariant under (13) to order g but not to order g^2. So we will add a second order in g term in the Lagrangian, say $g^2\mathcal{L}_{(2)}$, which would cancel the order g variation of $\mathcal{L}_1$. We have,

$$\begin{aligned}\delta\mathcal{L}_{(1)} &= \delta\left(-A_\mu j^\mu_{(0)}\right)\\ &= -\delta A_\mu j^\mu_{(0)} - A_\mu \delta j^\mu_{(0)}\\ &= \underbrace{\partial_\mu \alpha\, j^\mu_{(0)}}_{O(g^0)} + \underbrace{g\,\left(2A_\mu \partial^\mu \alpha\, \Phi^\dagger \Phi\right)}_{O(g^1)}\end{aligned}$$

Thus we set,

$$\begin{aligned}\delta\mathcal{L}_{(2)} &= -2A_\mu \partial^\mu \alpha\, \Phi^\dagger\Phi\\ &= \delta\left(A_\mu A^\mu\right)\Phi^\dagger\Phi\\ &= \delta\left(A_\mu A^\mu \Phi^\dagger\Phi\right)\end{aligned}$$

since $\delta\left(\Phi^\dagger\Phi\right) = 0$ under (13). Thus we identify,

$$\mathcal{L}_{(2)} = A_\mu A^\mu \Phi^\dagger \Phi.$$

The lagrangian up to second order in g is thus,

$$\mathcal{L}_{SED} = \mathcal{L}_{(0)} + g\,\mathcal{L}_{(1)} + g^2\mathcal{L}_{(2)}$$

5. Since $\delta\mathcal{L}_{(2)}$ *does not* have any term which is linear in g,

$$\begin{aligned}\delta\mathcal{L}_{(2)} &= 2\left(\delta A_\mu\right) A^\mu \Phi^\dagger\Phi + A_\mu A^\mu \delta(\Phi^\dagger\Phi)\\ &= O(g^0).\end{aligned}$$

So one does not need any higher order (in g) correction term! Thus the series of interaction terms ends at the second order, i.e. g^2. Thus the full Lagrangian is,

$$\begin{aligned}\mathcal{L}_{SED} &= \mathcal{L}_{(0)} + g\,\mathcal{L}_{(1)} + g^2\mathcal{L}_{(2)}\\ &= \left(\partial^\mu\Phi^\dagger\right)\left(\partial_\mu\Phi\right) - m^2\Phi^\dagger\Phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - g\,A_\mu j^\mu_{(0)} + g^2 A_\mu A^\mu \Phi^\dagger\Phi.\end{aligned} \tag{14}$$

Homework: Check that the lagrangian (14) leads to consistent equations for the scalar and Maxwell fields.

The moral of the story is that in order to consistently couple the complex scalar field (which has a global $U(1)$ symmetry) to the Maxwell gauge field, A_μ it is necessary to promote the global $U(1)$ symmetry to a ***local*** $U(1)$ symmetry by identifying the $U(1)$ symmetry parameter of the scalar field with the gauge symmetry parameter of the Maxwell field. This will turn out to be a general paradigm for all interactions, the only way to couple matter to massless fields (such as the Maxwell field or gluon field or the gravitational field) is via some gauge symmetry.

2.1 Principle of minimal coupling

If we look hard at the lagrangian (14), specifically the kinetic term scalar field and the interaction terms, it becomes clear that one can reorganize them in a nice way as follows

$$\left(\partial^\mu \Phi^\dagger\right)(\partial_\mu \Phi) - ig\, A_\mu \left(\Phi \partial^\mu \Phi^\dagger - \Phi^\dagger \partial^\mu \Phi\right) + g^2 A_\mu A^\mu \Phi^\dagger \Phi = (D^\mu \Phi)^\dagger (D_\mu \Phi)$$

where,

$$D_\mu \Phi \equiv (\partial_\mu - i\, g\, A_\mu)\, \Phi$$

Thus it appears if we replace the partial derivative, ∂_μ by a "covariant derivative" (or more precisely a gauge covariant derivative), $D_\mu = \partial_\mu - ig\, A_\mu$ in the non-interacting lagrangian, $\mathcal{L}_{(0)}$, then we automatically arrive at the consistently interacting lagrangian, $\mathcal{L}_{SED}$, (14),

$$\mathcal{L}_{SED} = (D^\mu \Phi)^\dagger (D_\mu \Phi) - m^2 \Phi^\dagger \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

This very simple and general recipe to construct consistently interacting theories of gauge fields (massless integer-spin fields) and matter, where we just replace the partial derivatives by a covariant derivative is called the principle of ***minimal substitution*** or ***minimal coupling.***