Fall 2019: Classical Field Theory (PH6297)

Hamiltonian Field Theory

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1 Introduction

So far we have defined classical field theory using a action principle by specifying the Lagrangian (density). This approach is called Lagrangian field theory and is best suited for quantizing a field theory using the path-integral (aka functional integral) approach. However, the more traditional route to quantizing a classical field theory is thru the "canonical quantization" program which is what you will find in most introductory treatments on quantum field theory. So for quantizing the field theory using the canonical quantization route, the Lagrangian is just a crutch to extract the Hamiltonian. However there is a price to pay when we switch from a Lagrangian approach to a Hamiltonian approach, that is we have to sacrifice manifest Lorentz invariance. In particular in the Hamiltonian approach time is treated differently from spatial coordinates. Recall that a field theory is a system of infinite degrees of freedom, a degree of freedom being located at each point in space, $\mathbf{x}$. If this degree of freedom is a Lorentz scalar, we will denote the degree of freedom located at position $\mathbf{x}$ by $\varphi(\mathbf{x})$ and its conjugate momenta, $\pi(\mathbf{x})$. These degrees of freedom of course fluctuate in time, so when we include this time-dependence, the notation becomes, $\varphi(\mathbf{x}, t)$ and $\pi(\mathbf{x}, t)$. The job of Hamiltonian field theory is describe time-evolution of π and φ in equations as follows,

$$\dot{\pi} = \ldots, \dot{\varphi} = \ldots$$

The dot is (total) time derivative, i.e. we have to choose a time axis or equivalently preferred reference frame which breaks manifest Lorentz invariance. Recall that earlier, in the Lagrangian/action approach we did not have to choose a time-frame because the equation of motion looked Lorentz invariant, e.g. the Euler-Lagrange equation for scalar field,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi \right)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0.$$

Here all Lorentz indices are contracted and the only quantities that appear are Lagrange density$\mathcal{L}$ and Φ which are both Lorentz invariants/scalars - so the whole equation is manifestly Lorentz invariant. In this note we review Hamiltonian field theory with an eye towards canonical quantization of classical fields. Since in most texts in classical mechanics, the dependence on time for the canonical pair of variables p and q are omitted from most formulas, we too shall choose not to display the time dependence, i.e. instead of $\varphi(\mathbf{x}, t)$ we will use, $\varphi(\mathbf{x})$ and instead of $\pi(\mathbf{x}, t)$ we will use, $\Pi(\mathbf{x})$. Just as in case of q, p it would be implicitly understood that $\varphi(\mathbf{x}), \pi(\mathbf{x})$ are functions of time.

2 Hamilton's principle and Hamilton's equation for field theory

Hamilton's principle for classical mechanics states that the equations of motion of a physical system described by the generalized coordinate and momenta, (q, p), can be obtain by extremizing (i.e. setting the first order variation to zero) of the following functional,

$$I[p, q] = \int dt \; [p\dot{q} - H(p, q)] . \tag{1}$$

The function, $H(p, q)$ is called the Hamiltonian function. One can easily check by varying the above action that the equation of motion for this system are,

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}, \\ \dot{p} &= -\frac{\partial H}{\partial q}. \end{aligned} \tag{2}$$

These equations are the well known Hamilton's equation. Note that in contrast with the Lagrangian approach where the Euler-Lagrange equations involve second order time-derivatives in time, the equations of motion in the Hamilton's approach are first order differential equations in time. Of course the price to pay is that we have twice the number of equations, one set for q and one set for p.

One can easily generalize Hamilton's framework for classical mechanics for a single degree of freedom to field theory i.e. infinite degrees of freedom. To accomplish that first we write down the action for N number of degrees of freedom, (q_i, p_i), $i = 1, 2, \ldots, N$. The action and Hamilton's equation for this case involving N degrees of freedom are:

$$I = \int dt \; \left[\left(\sum_{i=1}^{N} p_i \, \dot{q}_i \right) - H(p_i, q_i) \right] \tag{3}$$

and,

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}. \end{aligned} \tag{4}$$

Now in order to go to a field theory we have to take the continuum limit i.e. we set $N \to \infty$, then the discrete label, i turns into a continuum label such as the position coordinates, $\mathbf{x} = (x, y, z)$ (which are real number/continuous valued not discrete/integer valued),

$$\begin{aligned} q_i &\to \varphi(\mathbf{x}), \\ p_i &\to \pi(\mathbf{x}). \end{aligned}$$

The summation of course is turned into an integration over the continuum position coordinate,

$$\sum_{i=1}^{N} \to \int d^3\mathbf{x}.$$

Finally one should also realize that for a field theory, the Hamiltonian function would change into a functional of the fields, $\Phi(\mathbf{x}), \Pi(\mathbf{x})$,

$$H(p_i, q_i) \to H\left[\pi(\mathbf{x}), \varphi(\mathbf{x})\right].$$

In particular for a local field theory this functional is a spatial volume integral of a local density, $\mathcal{H}$

$$H\left[\pi(\mathbf{x}), \varphi(\mathbf{x})\right] = \int d^3\mathbf{x}\, \mathcal{H}(\pi(\mathbf{x}), \varphi(\mathbf{x})).$$

The local functional $\mathcal{H}$ is quite naturally called the Hamiltonian density since its volume integral is the Hamiltonian.

Making these changes to the Eq.s (3, 4), we get the the action for a Hamiltonian field theory,

$$I\left[\varphi(\mathbf{x},t), \pi(\mathbf{x},t)\right] = \int dt\, \left[\left(\int d^3\mathbf{x}\, \pi(\mathbf{x})\, \dot{\varphi}(\mathbf{x})\right) - H\left[\pi(\mathbf{x}), \varphi(\mathbf{x})\right]\right], \tag{5}$$

and by varying it, the Hamilton's equations for a field theory,

$$\begin{aligned} \dot{\varphi}(\mathbf{x}) &= \frac{\delta H}{\delta \pi(\mathbf{x})}, \\ \dot{\pi}(\mathbf{x}) &= -\frac{\delta H}{\delta \varphi(\mathbf{x})}. \end{aligned} \tag{6}$$

The new kind of derivative $\frac{\delta}{\delta\varphi(\mathbf{x})}$ is called a *functional derivative*. Roughly speaking it is the continuum limit or continuum version of a the notion of partial derivative for finite d.o.f. systems, i.e.

$$\frac{\partial}{\partial q_i} \to \frac{\delta}{\delta \varphi(\mathbf{x})}.$$

The rules of functional integration can be easily deduced by generalizing the following formula for $N-$degrees of freedom to continuum limit, namely,

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij}, \quad \frac{\partial p_i}{\partial p_j} = \delta_{ij}.$$

Using the prescribed rules, we now replace the i, j labels by position coordinates, $\mathbf{x}, \mathbf{y}$. To wit,

$$\frac{\delta \varphi(\mathbf{x})}{\delta \varphi(\mathbf{y})} = \delta^3(\mathbf{x}-\mathbf{y}), \quad \frac{\delta \pi(\mathbf{x})}{\delta \pi(\mathbf{y})} = \delta^3(\mathbf{x}-\mathbf{y}). \tag{7}$$

We will supplement the definition of the functional derivative (7), with two other rules:

1. The Chain rule of functional differentiation: For a function, $f(\mathbf{x}) \equiv f(\varphi(\mathbf{x}), \pi(\mathbf{y}))$,

$$\frac{\delta f(\mathbf{x})}{\delta \varphi(\mathbf{y})} = \frac{\partial f(\mathbf{x})}{\partial \varphi(\mathbf{x})} \frac{\delta \varphi(\mathbf{x})}{\delta \varphi(\mathbf{y})}, \tag{8}$$

   and,

$$\frac{\delta f(\mathbf{x})}{\delta \pi(\mathbf{y})} = \frac{\partial f(\mathbf{x})}{\partial \pi(\mathbf{x})} \frac{\delta \pi(\mathbf{x})}{\delta \pi(\mathbf{y})}. \tag{9}$$

e.g., $f(\mathbf{x}) = \varphi(\mathbf{x})^2\, \pi(\mathbf{x})$,

$$\frac{\delta\left(\varphi(\mathbf{x})^2\, \pi(\mathbf{x})\right)}{\delta\varphi(\mathbf{y})} = \frac{\partial\left(\varphi(\mathbf{x})^2\, \pi(\mathbf{x})\right)}{\partial\varphi(\mathbf{x})}\, \frac{\delta\varphi(\mathbf{x})}{\delta\varphi(\mathbf{y})}$$
$$= \quad 2\varphi(\mathbf{x}\,)\pi(\mathbf{x})\, \frac{\delta\varphi(\mathbf{x})}{\delta\varphi(\mathbf{y})}.$$

2. Functional differentiation commutes with spatial partial derivatives or spatial integration,

$$\frac{\delta}{\delta\varphi(\mathbf{x})}\frac{\partial\,(\ldots)}{\partial y^i} = \frac{\partial}{\partial y^i}\frac{\delta\,(\ldots)}{\delta\varphi(\mathbf{x})}, \tag{10}$$

$$\frac{\delta}{\delta\varphi(\mathbf{x})}\int d^3\mathbf{y}\;(\ldots) = \int d^3\mathbf{y}\;\frac{\delta}{\delta\varphi(\mathbf{x})}\,(\ldots)\,. \tag{11}$$

3 Hamiltonian from the Lagrangian: Legendre Transformations

Starting with a Lagrangian for a physical system, one can extract the Hamiltonian thru a Legendre transform. Two functions, $f(x)$ and $g(x)$ are said to be Legendre transforms of each other if their first derivatives are functional inverse of each other, i.e.

$$x = f'(g'(x)) = \left.\frac{df(y)}{dy}\right|_{y=g'(x)}$$

where primes denote first derivative of the function wrt their respective arguments. Solving this condition one gets,

$$x\, g'(x) = f(y)$$

So the Hamiltonian is defined to be the Legendre transform of the Lagrangian,

$$H(p,q) = p\,\dot{q}(p) - L(q,\dot{q}(p)).$$

Now here $\dot{q}(p)$ means we have inverted the relation,

$$p = \frac{\partial L(q,\dot{q})}{\partial\dot{q}}$$

to express, $\dot{q}$ as a function of p (and more generally a function of both p and q). For a system with N-degrees of freedom, the definition of momentum conjugate to q_i is,

$$p_i = \frac{\partial L}{\partial\dot{q}_i}.$$

Inverting this[1] we will get velocities in terms of momenta,

$$\dot{q}_i = \dot{q}_i\,(p_j, q_j)\,.$$

[1]This is only allowed when the Hessian matrix, which is the Jacobian of transformation from velocities to momenta is invertible, i.e. the determinant is non-singular

$$\left|\frac{\partial p_i}{\partial\dot{q}_j}\right| = \left|\frac{\partial^2 L}{\partial\dot{q}_i\,\partial\dot{q}_j}\right| \neq 0.$$

Then we can perform a multivariable Legendre transform to get the Hamiltonian,

$$H = \left(\sum_i p_i \, \dot{q}_i \right) - L(q_i, \dot{q}_i).$$

Now we can follow the prescription mentioned in the previous section to take the continuum limit, $N \to \infty$ whereby we obtain the Hamiltonian for a field theory from the Lagrangian. In field theory, the Lagrangian itself can be expressed as a volume integral of Lagrangian density,

$$L[\varphi(\mathbf{x}), \dot{\varphi}(\mathbf{x})] = \int d^3\mathbf{x} \mathcal{L}, \; \mathcal{L} = \mathcal{L}(\varphi(\mathbf{x}), \dot{\varphi}(\mathbf{x}), \boldsymbol{\nabla}\varphi(\mathbf{x})).$$

From this we derive the momenta,

$$\pi(\mathbf{x}) = \frac{\delta L}{\delta \dot{\varphi}(\mathbf{x})},$$

and inverting this we express $\dot{\varphi}(\mathbf{x})$ in terms of $\pi(\mathbf{x})$ (and more generally both $\pi(\mathbf{x})$ as well as $\varphi(\mathbf{x})$). Then we get the Hamiltonian,

$$\begin{aligned} H &= \left(\int d^3\mathbf{x} \, \pi(\mathbf{x}) \, \dot{\varphi}(\mathbf{x}) \right) - L[\varphi(\mathbf{x}), \dot{\varphi}(\mathbf{x})] \\ &= \int d^3\mathbf{x} \, \left(\pi(\mathbf{x}) \, \dot{\varphi}(\mathbf{x}) - \mathcal{L} \right). \end{aligned}$$

Thus the Hamiltonian density is

$$\mathcal{H} = \pi(\mathbf{x}) \, \dot{\varphi}(\mathbf{x}) - \mathcal{L},$$

where $\dot{\varphi} = \dot{\varphi}(\pi)$.

3.1 Example: Real (free) scalar field theory

The real scalar field theory is defined by the action,

$$I[\varphi(x)] = \int d^4x \; \mathcal{L}, \; \mathcal{L} = \frac{1}{2} \left(\partial_\mu \varphi\right) \left(\partial^\mu \varphi\right) - \frac{1}{2} m^2 \varphi^2.$$

So the Lagrangian is,

$$\begin{aligned} L &= \int d^3\mathbf{x} \, \left(\frac{1}{2} \left(\partial_\mu \varphi\right) \left(\partial^\mu \varphi\right) - \frac{1}{2} m^2 \varphi^2 \right), \\ &= \int d^3\mathbf{x} \, \left(\frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\varphi - \frac{1}{2} m^2 \varphi^2 \right). \end{aligned}$$

As expected the Lagrangian is a functional of the coordinates and velocities, namely, $\varphi(\mathbf{x})$ and $\dot{\varphi}(\mathbf{x})$ respectively. The momentum conjugate to $\varphi(x)$ is,

$$\begin{aligned}
\pi(\mathbf{x}) &= \frac{\delta L}{\delta \dot{\varphi}(\mathbf{x})} \\
&= \frac{\delta}{\delta \dot{\varphi}(\mathbf{x})} \int d^3\mathbf{y} \, \left(\underline{\frac{1}{2}\dot{\varphi}^2(\mathbf{y})} - \frac{1}{2}\boldsymbol{\nabla}_{\mathbf{y}}\varphi(\mathbf{y}) \cdot \boldsymbol{\nabla}_{\mathbf{y}}\varphi(\mathbf{y}) - \frac{1}{2}m^2\varphi^2(\mathbf{y}) \right) \\
&= \int d^3\mathbf{y} \, \frac{\delta}{\delta \dot{\varphi}(\mathbf{x})} \left(\frac{1}{2}\dot{\varphi}^2(\mathbf{y}) \right) \\
&= \int d^3\mathbf{y} \, \dot{\varphi}(\mathbf{y}) \, \frac{\delta \dot{\varphi}(\mathbf{y})}{\delta \dot{\varphi}(\mathbf{x})} \\
&= \int d^3\mathbf{y} \, \dot{\varphi}(\mathbf{y}) \, \delta^3(\mathbf{x} - \mathbf{y}) \\
&= \dot{\varphi}(\mathbf{x}).
\end{aligned}$$

We can use this to eliminate $\dot{\varphi}$ in terms of π,

$$\dot{\varphi} = \pi.$$

In terms of the momenta, the Lagrangian then becomes,

$$L = \int d^3\mathbf{x} \, \left(\frac{1}{2}\pi^2 - \frac{1}{2}\boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\varphi - \frac{1}{2}m^2\varphi^2 \right),$$

and the Hamiltonian is,

$$\begin{aligned}
H &= \left(\int d^3\mathbf{x} \, \pi(\mathbf{x}) \, \dot{\varphi}(\mathbf{x}) \right) - L \\
&= \left(\int d^3\mathbf{x} \, \pi^2(\mathbf{x}) \right) - L \\
&= \left(\int d^3\mathbf{x} \, \pi^2(\mathbf{x}) \right) - \int d^3\mathbf{x} \, \left(\frac{1}{2}\pi^2 - \frac{1}{2}\boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\varphi - \frac{1}{2}m^2\varphi^2 \right) \\
&= \int d^3\mathbf{x} \, \left(\frac{1}{2}\pi^2 + \frac{1}{2}\boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\varphi + \frac{1}{2}m^2\varphi^2 \right). \qquad (12)
\end{aligned}$$

The Hamiltonian density is clearly,

$$\mathcal{H} = \frac{1}{2}\pi^2 + + \frac{1}{2}\boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\varphi + \frac{1}{2}m^2\varphi^2.$$

From the expression for the Hamiltonian (12), we can extract the Hamilton's equations. The first Hamilton's equation doesn't give us anything but the old relation,

$$\begin{aligned}
\dot{\varphi}(\mathbf{x}) &= \frac{\delta H}{\delta \pi(\mathbf{x})} \\
&= \pi(\mathbf{x}). \qquad (13)
\end{aligned}$$

However the second equation is non-trivial,,

$$
\begin{aligned}
-\dot{\pi}(\mathbf{x}) &= \frac{\delta H}{\delta\varphi(\mathbf{x})} \\
&= \frac{\delta}{\delta\varphi(\mathbf{x})}\int d^3\mathbf{y}\,\left(\frac{1}{2}\pi^2 + \underline{\frac{1}{2}\nabla_{\mathbf{y}}\varphi\cdot\nabla_{\mathbf{y}}\varphi + \frac{1}{2}m^2\varphi^2}\right) \\
&= \int d^3\mathbf{y}\,\left[\nabla_{\mathbf{y}}\varphi\cdot\nabla_{\mathbf{y}}\left(\frac{\delta}{\delta\varphi(\mathbf{x})}\varphi(\mathbf{y})\right) + m^2\varphi(\mathbf{y})\,\frac{\delta\varphi(\mathbf{y})}{\delta\varphi(\mathbf{x})}\right] \\
&= \int d^3\mathbf{y}\,\left[\nabla_{\mathbf{y}}\varphi\cdot\nabla_{\mathbf{y}}\left(\delta^3(\mathbf{x}-\mathbf{y})\right) + m^2\varphi(\mathbf{y})\,\delta^3(\mathbf{x}-\mathbf{y})\right] \\
&= \int d^3\mathbf{y}\,\left(-\nabla_{\mathbf{y}}^2\varphi + m^2\varphi\right)\,\delta^3(\mathbf{x}-\mathbf{y}) \\
&= -\left(\nabla^2 - m^2\right)\varphi(\mathbf{x}).
\end{aligned}
\tag{14}
$$

So equations are,

$$
\dot{\varphi}(\mathbf{x}) = \pi(\mathbf{x}), \qquad \dot{\pi}(\mathbf{x}) = \left(\nabla^2 - m^2\right)\varphi(\mathbf{x}).
$$

Replacing, $\pi = \dot{\varphi}$ in the second equation we get the second equation to look like,

$$
\begin{aligned}
\ddot{\varphi} &= \left(\nabla^2 - m^2\right)\varphi, \\
\left(\partial_t^2 - \nabla^2 + m^2\right)\varphi &= 0.
\end{aligned}
$$

This is nothing but the well-familiar Klein-Gordon equation,

$$
\left(\Box + m^2\right)\Phi = 0.
$$

Thus we have recovered the correct equation of motion using the field Hamiltonian, Eq. (12) and the Hamilton's equations Eq. (6).

4 Poisson brackets, Charges and Algebra of charges

In general, the time evolution of a quantity, $f(p,q)$ can be deduced from the Poisson brackets (PB) of that quantity with the Hamiltonian, H,

$$
\begin{aligned}
\frac{df(p,q)}{dt} &= \frac{\partial f}{\partial q}\underline{\frac{dq}{dt}} + \frac{\partial f}{\partial p}\underbrace{\frac{dp}{dt}} \\
&= \frac{\partial f}{\partial q}\underline{\frac{\partial H}{\partial p}} - \frac{\partial f}{\partial p}\underbrace{\frac{\partial H}{\partial q}} \\
&\equiv [f,H]_{PB}\,.
\end{aligned}
$$

In particular if the system has conserved charges, Q, since, $\frac{dQ}{dt} = 0$, one has for conserved charge, Q,

$$
\{Q,H\} = 0.
$$

(However if the quantity f has an *explicit* dependence on t, i.e. $f = f(p,q,t)$ then the time evolution of f is given by,

$$
\frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t}. \tag{15}
$$

Similarly, if the expression of some charge has an explicit time dependence, the conservation law will look more like $\{Q, H\} + \frac{\partial Q}{\partial t} = 0$).

More generally for two quantities, $A(p, q)$ and $B(p, q)$ one can define a Poisson bracket,

$$[A, B]_{PB} \equiv \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}.$$

Evidently, the Poisson bracket is antisymmetric

$$[A, B]_{PB} = -[B, A]_{PB}.$$

In particular we have the canonical Poisson bracket (algebra)

$$[q, q]_{PB} = 0 = [p, p]_{PB}$$

$$[q, p]_{PB} = 1.$$

Next we generalize the Poisson bracket expression for single degree of freedom to field theory to N degrees of freedom. For N degrees of freedom, the Poisson bracket can be easily shown to be of the form,

$$[A, B]_{PB} \equiv \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right).$$

In this case the canonical Poisson bracket algebra is,

$$[q_i, q_j]_{PB} = 0 = [p_i, p_j]_{PB}$$

$$[q_i, p_j]_{PB} = \delta_{ij}.$$

Finally, to go to field theory, we have to take the continuum limit, $N \to \infty$, we replace, $q_i, p_i \to \Phi(\mathbf{x}), \Pi(\mathbf{x})$ and $\sum_i \to \int d^3\mathbf{x}$, and get,

$$[A, B]_{PB} \equiv \int d^3\mathbf{x} \left(\frac{\delta A}{\delta\varphi(\mathbf{x})}\frac{\delta B}{\delta\pi(\mathbf{x})} - \frac{\delta A}{\delta\pi(\mathbf{x})}\frac{\delta B}{\delta\varphi(\mathbf{x})}\right). \tag{16}$$

Here of course A, B are themselves functionals of φ, π, i.e. $A, B = A, B\,[\varphi(\mathbf{x}), \pi(\mathbf{x})]$. The canonical Poisson brackets for classical field theory are then,

$$[\varphi(\mathbf{x}), \varphi(\mathbf{y})]_{PB} = 0 = [\pi(\mathbf{x}), \pi(\mathbf{y})]_{PB}. \tag{17}$$

$$[\varphi(\mathbf{x}), \pi(\mathbf{y})]_{PB} = \delta^3(\mathbf{x} - \mathbf{y}). \tag{18}$$

4.0.1 Sidebar on Canonical Quantization

It is very simple to quantize the classical field theory once we have the Poisson bracket (16) and the canonical Poisson bracket algebra (17 & 18). This done by first promoting the the fields, φ, π (and any functionals of them such as A and B) to field operators,

$$\varphi(\mathbf{x}) \to \hat{\varphi}(\mathbf{x}), \qquad \pi(\mathbf{x}) \to \hat{\pi}(\mathbf{x})$$
$$A\left[\varphi(\mathbf{x}), \pi(\mathbf{x})\right] \to A\left[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{x})\right].$$

And then one replaces the Poisson brackets by the commutator brackets,

$$[A, B]_{PB} \to \frac{1}{i}\,[A, B]\,.$$

In particular the canonical Poisson bracket algebra (17 & 18) becomes in quantum theory the following commutator algebra,

$$[\varphi(\mathbf{x}), \varphi(\mathbf{y})] = 0 = [\pi(\mathbf{x}), \pi(\mathbf{y})]\,,$$
$$[\varphi(\mathbf{x}), \pi(\mathbf{y})] = i\,\delta^3(\mathbf{x} - \mathbf{y}).$$

This prescription to derive a quantum theory from the classical theory (Poisson brackets and the canonical Poisson bracket algebra) is called ***canonical quantization.*** It is easy to see that if we follow this prescription of replacing the Poisson bracket by the commutator bracket (divided by i), the Hamilton's equation for a general (phase space) function f (15) becomes the Heisenberg equation of the motion for the operator f.

$$\frac{d\hat{f}}{dt} = \frac{1}{i}\left[\hat{f}, \hat{H}\right] + \frac{\partial \hat{f}}{\partial t}.$$

4.1 Example: Free Scalar field and Conserved charges

We already know thru Noether's theorem that there are charges corresponding to Poincare symmetry, namely the energy-momenta, (P^0, P^i) and the boost-rotation charges $(L^{\mu\nu})$. Let's check that these are conserved by taking the Poisson bracket with the Hamiltonian. First we need to express the charges in the canonical variables, φ, π i.e. by eliminating the velocity, $\dot{\varphi}$ in lieu of the conjugate momenta, $\pi(\mathbf{x})$. We start from the expression,

$$\begin{aligned} P^\mu &= \int d^3\mathbf{x}\, T^{0\mu} \\ &= \int d^3\mathbf{x}\, \left[\dot{\varphi}(\mathbf{x})\,\partial^\mu\varphi(\mathbf{x}) - \eta^{0\mu}\mathcal{L}(\mathbf{x})\right]. \end{aligned} \tag{19}$$

$$\begin{aligned} \Longrightarrow P^0 &= \int d^3\mathbf{x}\left[\dot{\varphi}^2(\mathbf{x}) - \mathcal{L}\right] \\ &= \int d^3\mathbf{x}\left[\dot{\varphi}^2(\mathbf{x}) - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{m^2}{2}\varphi^2\right] \\ &= \int d^3\mathbf{x}\left[\frac{1}{2}\dot{\varphi}^2(\mathbf{x}) + \frac{1}{2}\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\varphi + \frac{m^2}{2}\varphi^2\right] \\ &= \int d^3\mathbf{x}\left[\frac{1}{2}\pi^2(\mathbf{x}) + \frac{1}{2}\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\varphi + \frac{m^2}{2}\varphi^2\right], \end{aligned} \tag{20}$$

Notice that, $P^0 = H$! This is no surprise because P^0 is the energy and the Hamiltonian for conservative systems also correspond to energy. Let's check if it is conserved or not. According to the equation of motion,

$$\frac{dP^0}{dt} = \left[P^0, H\right]_{PB}$$

But since $P^0 = H$, we have $\left[P^0, H\right]_{PB} = [H,H]_{PB} = 0$. Thus the rhs of this equation vanishes and we have the energy to be conserved, $\frac{dP^0}{dt} = 0$.
Next, let's look at the linear momenta by taking $\mu = i$ in (19),

$$\begin{aligned} P^i &= \int d^3\mathbf{x}\ \left[\dot\varphi(\mathbf{x})\,\partial^i\varphi(\mathbf{x}) - \eta^{0i}\mathcal{L}\right] \\ &= \int d^3\mathbf{x}\,\dot\varphi(\mathbf{x})\,\partial^i\varphi(\mathbf{x}) \\ &= -\int d^3\mathbf{x}\,\partial_i\varphi(\mathbf{x})\,\pi(\mathbf{x}). \end{aligned} \tag{21}$$

Here, we have used that due to the fact that the metric component, $\eta_{ii} = -1$, $\partial^i = -\partial_i$; and we have replaced $\dot\varphi = \pi$.

As in the case of P^0 we have the time-evolution law, $\frac{dP^i}{dt} = \left[P^i, H\right]_{PB}$. So we need to evaluate,

$$\left[P^i, H\right]_{PB} = \int d^3\mathbf{y}\ \left(\frac{\delta P^i}{\delta\varphi(\mathbf{y})}\,\frac{\delta H}{\delta\pi(\mathbf{y})} - \frac{\delta P^i}{\delta\pi(\mathbf{y})}\,\frac{\delta H}{\delta\varphi(\mathbf{y})}\right). \tag{22}$$

From Eq. (21) and using chain rules (8, 9) we compute

$$\begin{aligned} \frac{\delta P^i}{\delta\varphi(\mathbf{y})} &= \frac{\delta}{\delta\varphi(\mathbf{y})}\left(-\int d^3\mathbf{x}\,\partial_i\varphi(\mathbf{x})\,\pi(\mathbf{x})\right) \\ &= -\int d^3\mathbf{x}\,\frac{\delta\left(\partial_i\varphi(\mathbf{x})\right)}{\delta\varphi(\mathbf{y})}\,\pi(\mathbf{x}) \\ &= -\int d^3\mathbf{x}\,\partial_i\left(\frac{\delta\varphi(\mathbf{x})}{\delta\varphi(\mathbf{y})}\right)\,\pi(\mathbf{x}) \\ &= -\int d^3\mathbf{x}\,\partial_i\left(\delta^3(\mathbf{x}-\mathbf{y})\right)\,\pi(\mathbf{x}) \\ &= \int d^3\mathbf{x}\,\delta^3(\mathbf{x}-\mathbf{y})\,\partial_i\pi(\mathbf{x}) \\ &= \frac{\partial\pi(\mathbf{y})}{\partial y^i}, \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\delta P^i}{\delta\pi(\mathbf{y})} &= -\frac{\delta}{\delta\pi(\mathbf{y})}\int d^3\mathbf{x}\,\partial_i\varphi(\mathbf{x})\,\pi(\mathbf{x}) \\ &= -\int d^3\mathbf{x}\,\partial_i\varphi(\mathbf{x})\,\frac{\delta}{\delta\pi(\mathbf{y})}\pi(\mathbf{x}) \\ &= -\int d^3\mathbf{x}\,\partial_i\varphi(\mathbf{x})\,\delta^3(\mathbf{x}-\mathbf{y}) \\ &= -\frac{\partial\varphi(\mathbf{y})}{\partial y^i}. \end{aligned} \tag{24}$$

Now recall from Eq. (13) and (14),

$$\frac{\delta H}{\delta\varphi(\mathbf{y})} = -\left(\boldsymbol{\nabla}_{\mathbf{y}}^2 - m^2\right)\varphi(\mathbf{y}).,\qquad \frac{\delta H}{\delta\pi(\mathbf{y})} = \pi(\mathbf{y}). \tag{25}$$

Now plugging Eq. (23), (24) and (25) in the expression for the PB in RHS of Eq. (22), we obtain,

$$\begin{aligned}\left[P^i, H\right]_{PB} &= \int d^3\mathbf{y}\left(\frac{\delta P^i}{\delta\varphi(\mathbf{y})}\frac{\delta H}{\delta\pi(\mathbf{y})} - \frac{\delta P^i}{\delta\pi(\mathbf{y})}\frac{\delta H}{\delta\varphi(\mathbf{y})}\right)\\ &= \int d^3\mathbf{y}\left(\frac{\partial}{\partial y^i}\pi(\mathbf{y})\,\pi(\mathbf{y}) - \frac{\partial}{\partial y^i}\varphi(\mathbf{y})\,\left(\boldsymbol{\nabla}_{\mathbf{y}}^2 - m^2\right)\varphi(\mathbf{y})\right)\\ &= \int d^3\mathbf{y}\left(\frac{\partial}{\partial y^i}\pi(\mathbf{y})\,\pi(\mathbf{y}) - \frac{\partial}{\partial y^i}\varphi(\mathbf{y})\,\boldsymbol{\nabla}_{\mathbf{y}}^2\varphi - m^2\frac{\partial}{\partial y^i}\varphi(\mathbf{y})\varphi(\mathbf{y})\right)\\ &= \int d^3\mathbf{y}\,\frac{\partial}{\partial y^i}\left(\frac{1}{2}\pi^2(\mathbf{y}) + \frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2 + \frac{m^2}{2}\varphi^2(\mathbf{y})\right) + \int d^3\mathbf{y}\,\nabla\left(\partial_i\varphi\boldsymbol{\nabla}\varphi\right),\end{aligned} \tag{26}$$

where we have used,

$$\begin{aligned}\partial_i\varphi\boldsymbol{\nabla}^2\varphi &= \partial_i\varphi\partial_j\partial_j\varphi\\ &= \partial_j\left(\partial_i\varphi\partial_j\varphi\right) - \partial_j\left(\partial_i\varphi\right)\partial_j\varphi\\ &= \partial_j\left(\partial_i\varphi\partial_j\varphi\right) - \partial_i\partial_j\varphi\,\partial_j\varphi\\ &= \partial_j\left(\partial_i\varphi\partial_j\varphi\right) - \partial_i\left(\frac{1}{2}\partial_j\varphi\partial_j\varphi\right)\\ &= \boldsymbol{\nabla}\cdot\left(\partial_i\varphi\boldsymbol{\nabla}\varphi\right) - \partial_i\left(\frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2\right).\end{aligned}$$

Thus both terms in the expression for $\left[P^i, H\right]_{PB}$ i.e. the rhs of (26) are integral of a total derivative and after integration turns into surface terms at spatial infinity where of course they are zero (we are assuming for all integrals to be finite that all fields decay to zero at spatial infinity). Thus

$$\left[P^i, H\right]_{PB} = 0$$

which means P^i i.e. the linear momentum is conserved as well,

$$\frac{dP^i}{dt} = 0.$$

For the boost and rotation symmetry combined i.e. Lorentz symmetry, the Noether charges are given by the expression,

$$\begin{aligned}L^{\mu\nu} &= \int d^3\mathbf{x}\,M^{0\mu\nu}\\ &= \int d^3\mathbf{x}\,\left(x^\mu\Theta^{0\nu} - x^\nu\Theta^{0\mu}\right),\end{aligned}$$

In particular, these charges are, the angular momenta,

$$\begin{aligned}L^{ij} &= \int d^3\mathbf{x}\,\left(x^i\Theta^{0j} - x^j\Theta^{0i}\right)\\ &= \int d^3\mathbf{x}\,\left(-x^i\partial_j\varphi(\mathbf{x})\,\pi(\mathbf{x}) + x^j\partial_i\varphi(\mathbf{x})\,\pi(\mathbf{x})\right).\end{aligned}$$

and the less familiar "Boost charges" or "Boost generators"

$$
\begin{aligned}
L^{0i} &= \int d^3\mathbf{x}\ \left(x^0\Theta^{0i} - x^i\Theta^{00}\right) \\
&= \int d^3\mathbf{x}\ \left(t\,\Theta^{0i} - x^i\,\mathcal{H}\right), \\
&= t\,P^i - \int d^3\mathbf{x}\,x^i\,\mathcal{H}
\end{aligned}
$$

Since this is a conserved charge, its value at any time should be the same as at time, $t = 0$. So setting $t = 0$ in the above expression,

$$
L^{0i} = -\int d^3\mathbf{x}\,x^i\,\mathcal{H}|_{t=0}.
$$

This can be thought of as a "moment of energy" or "angular energy".

Lets now check that the angular momentum is conserved i.e. $\left[L^{ij}, H\right]_{PB} = 0$. To compute this Poisson bracket we first evaluate the following functional derivatives,

$$
\begin{aligned}
\frac{\delta L^{ij}}{\delta\varphi(\mathbf{x})} &= \frac{\delta}{\delta\varphi(\mathbf{x})}\int d^3\mathbf{y}\ \left(-y^i\frac{\partial\varphi(\mathbf{y})}{\partial y^j}\,\pi(\mathbf{y}) + y^j\frac{\partial\varphi(\mathbf{y})}{\partial y^i}\,\pi(\mathbf{y})\right) \\
&= \int d^3\mathbf{y}\ \left(-y^i\frac{\partial}{\partial y^j}\frac{\delta\varphi(\mathbf{y})}{\delta\varphi(\mathbf{x})}\,\pi(\mathbf{y}) + y^j\frac{\partial}{\partial y^i}\frac{\delta\varphi(\mathbf{y})}{\delta\varphi(\mathbf{x})}\,\pi(\mathbf{y})\right) \\
&= \int d^3\mathbf{y}\ \left(-y^i\frac{\partial}{\partial y^j}\delta^3(\mathbf{x}-\mathbf{y})\,\pi(\mathbf{y}) + y^j\frac{\partial}{\partial y^i}\delta^3(\mathbf{x}-\mathbf{y})\,\pi(\mathbf{y})\right) \\
&= \int d^3\mathbf{y}\ \delta^3(\mathbf{x}-\mathbf{y})\left[\frac{\partial}{\partial y^j}\left(y^i\,\pi(\mathbf{y})\right) - \frac{\partial}{\partial y^i}\left(y^j\,\pi(\mathbf{y})\right)\right] \\
&= \partial_j\left(x^i\,\pi(\mathbf{x})\right) - \partial_i\left(x^j\,\pi(\mathbf{x})\right) \\
&= x^i\partial_j\pi(\mathbf{x}) - x^j\partial_i\pi(\mathbf{x}). \qquad (27)
\end{aligned}
$$

and,

$$
\begin{aligned}
\frac{\delta L^{ij}}{\delta\pi(\mathbf{x})} &= \frac{\delta}{\delta\pi(\mathbf{x})}\int d^3\mathbf{y}\ \left(-y^i\frac{\partial\varphi(\mathbf{y})}{\partial y^j}\,\pi(\mathbf{y}) + y^j\frac{\partial\varphi(\mathbf{y})}{\partial y^i}\,\pi(\mathbf{y})\right) \\
&= \int d^3\mathbf{y}\ \left(-y^i\frac{\partial\varphi(\mathbf{y})}{\partial y^j}\,\frac{\delta\pi(\mathbf{y})}{\delta\pi(\mathbf{x})} + y^j\frac{\partial\varphi(\mathbf{y})}{\partial y^i}\,\frac{\delta\pi(\mathbf{y})}{\delta\pi(\mathbf{x})}\right) \\
&= \int d^3\mathbf{y}\ \left(-y^i\frac{\partial\varphi(\mathbf{y})}{\partial y^j}\,\delta^3(\mathbf{x}-\mathbf{y}) + y^j\frac{\partial\varphi(\mathbf{y})}{\partial y^i}\,\delta^3(\mathbf{x}-\mathbf{y})\right) \\
&= -x^i\partial_j\varphi(\mathbf{x}) + x^j\partial_i\varphi(\mathbf{x}). \qquad (28)
\end{aligned}
$$

Finally we compute the Poisson brackets by using Eq.s (25), (27) and (28)

$$\begin{aligned}
\left[L^{ij}, H\right]_{PB} &= \int d^3\mathbf{x}\, \left(\frac{\delta L^{ij}}{\delta\varphi(\mathbf{x})}\,\frac{\delta H}{\delta\pi(\mathbf{x})} - \frac{\delta L^{ij}}{\delta\pi(\mathbf{x})}\,\frac{\delta H}{\delta\varphi(\mathbf{x})}\right) \\
&= \int d^3\mathbf{x}\, \left[\left(x^i\partial_j\pi(\mathbf{x}) - x^j\partial_i\pi(\mathbf{x})\right)\pi(\mathbf{x}) - \left(x^i\partial_j\varphi(\mathbf{x}) - x^j\partial_i\varphi(\mathbf{x})\right)\left(\boldsymbol{\nabla}^2 - m^2\right)\varphi(\mathbf{x})\right] \\
&= \int d^3\mathbf{x}\, \left[\left(x^i\partial_j - x^j\partial_j\right)\left(\frac{\pi^2(\mathbf{x})}{2} + \frac{m^2\varphi^2(\mathbf{x})}{2}\right) - \left(x^i\partial_j\varphi(\mathbf{x}) - x^j\partial_i\varphi(\mathbf{x})\right)\boldsymbol{\nabla}^2\varphi(\mathbf{x})\right] \\
&= \int d^3\mathbf{x}\, \left[\partial_j\left(x^i\,\frac{\pi^2(\mathbf{x})}{2} + x^i\,\frac{m^2\varphi^2(\mathbf{x})}{2}\right) - \partial_i\left(x^j\,\frac{\pi^2(\mathbf{x})}{2} + x^j\,\frac{m^2\varphi^2(\mathbf{x})}{2}\right)\right. \\
&\qquad\left. - \left(x^i\partial_j\varphi(\mathbf{x}) - x^j\partial_i\varphi(\mathbf{x})\right)\boldsymbol{\nabla}^2\varphi(\mathbf{x})\right]. \qquad (29)
\end{aligned}$$

The first two terms are of course integrals of total derivatives which would lead to surface terms at spatial infinity which in turn vanish on imposing the proper boundary conditions. The last term needs to be massaged a little bit to convert it to a total derivative . We have already seen that,

$$\partial_i\varphi\boldsymbol{\nabla}^2\varphi = \boldsymbol{\nabla}\cdot\left(\partial_i\varphi\boldsymbol{\nabla}\varphi\right) - \partial_i\left(\frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2\right).$$

So then,

$$\begin{aligned}
\left(x^i\partial_j\varphi - x^j\partial_i\varphi\right)\boldsymbol{\nabla}^2\varphi &= x^i\,\boldsymbol{\nabla}\cdot\left(\partial_j\varphi\boldsymbol{\nabla}\varphi\right) - x^j\,\boldsymbol{\nabla}\cdot\left(\partial_i\varphi\boldsymbol{\nabla}\varphi\right) - x^i\partial_j\left(\frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2\right) + x^j\partial_i\left(\frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2\right) \\
&= x^i\partial_k\left(\partial_j\varphi\partial_k\varphi\right) - x^j\partial_k\left(\partial_i\varphi\partial_k\varphi\right) - x^i\partial_j\left(\frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2\right) + x^j\partial_i\left(\frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2\right) \\
&= \partial_k\left(x^i\,\partial_j\varphi\partial_k\varphi - x^j\partial_i\varphi\partial_k\varphi\right) - \partial_j\left(x^i\frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2\right) + \partial_i\left(x^j\frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2\right).
\end{aligned}$$

This is also a total derivative. Thus all three terms in the rhs of (29) are integrals of total derivatives which vanish on integration,

$$\left[L^{ij}, H\right]_{PB} = 0 \implies \frac{dL^{ij}}{dt} = 0.$$

Homework: Using this Poisson Bracket method, check that the boost charge, L^{0i} is also conserved i.e. $\frac{dL^{0i}}{dt} = 0$.