Mathematical Methods for Physicists, FK7048, fall 2018
Exercise sheet 1, **due Tuesday september 10th, 15:00**

Total amount of points: 12p, and 2 bonus points (2bp).

1 Barometric formula

Hydrostatics is governed by the following equation:

$$-\vec{\nabla} P + \rho\, \vec{g} = \vec{0} \tag{1}$$

where $\vec{g}$ is the gravitational acceleration, assumed constant, P and ρ are the pressure and density fields. A barotropic fluid is a fluid whose density depends only on pressure *viz.* $\rho = \rho(P)$.

(a) (1p) The earth's atmosphere can be considered in a first approximation as a barotropic fluid at equilibrium in the gravitational field $\vec{g}$. Find an integral relation from which the evolution of the pressure as a function of altitude z for any given relation $\rho(P)$ can be obtained. Consider only the vertical direction and do not forget to set the initial condition.

(b) (1p) Find the relation $\rho(P)$ for an ideal gas, starting from the usual equation of states (that is, the ideal gas las). Any parameter you may need to introduce will be considered constant so far.

(c) (1p+1bp) Use the expression $\rho(P)$ for an ideal gas in your solution of question (a) to find $P(z)$. You should obtain the so-called barometric formula $P(z) = P_0\, e^{-\frac{z}{h}}$. Give the analytical expression for h.
Bonus: obtain a numerical value for h. Use your physical intuition to evaluate the parameters you have introduced in question (b). Is the answer you obtain resonable?

(d) (1p) So far we have assumed that Earth's atmosphere is isothermal, however the temperature lapse rate is clearly non-negligible on the kilometer scale. What does the barometric formula become when choosing a linear profile $T(z) = T_0 - \alpha z$ for the temperature of the ideal gas?

2 Parallel RLC circuit

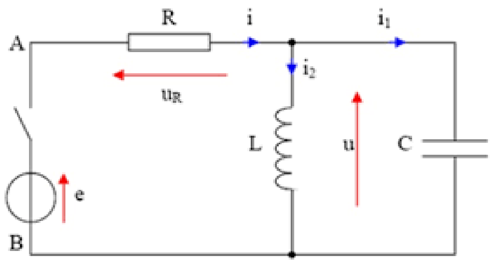

Figure 1: Parallel RLC circuit

Let us consider a resistor R, an inductor L and a capacitor C in parallel (see figure 1) with a sinusoisal forcing $e(t) = E\sin(\omega t)$.

(a) (1bp) Bonus question: Show that the voltage $u(t)$ of the capacitor satisfies the following 2nd order linear ODE:

$$\frac{d^2u}{dt^2} + \frac{1}{RC}\frac{du}{dt} + \frac{1}{LC}u(t) = \frac{1}{RC}\frac{de}{dt} \tag{2}$$

Set $\omega_0^2 = \frac{1}{LC}$ and $2\ \omega_0\ \sigma = \frac{1}{RC}$, with real $\omega_0 > 0$ and $\sigma \geq 0$

(b) (2p) Find the general solution for the homogeneous equation for all possible values of the parameter σ.

(c) (1p) To obtain the full solution of the ODE, find a particular solution. Hint: use $\Re\big(e^{i\omega t}\big) = \cos(\omega t)$ and look for solutions of the form 'polynomial times complex exponential', but keep in mind that the final result must be real.

(d) (1p) Physically describe what happens when $\omega = \omega_0$.

3 Thickness distribution of sea ice

Background:
Consider a large area $\mathscr{A}$ in the Arctic region. Within this area there is a surface $A(h_1, h_2)$ covered by sea ice of thickness h in the range $[h_1, h_2]$. Then, the thickness distribution $g(h,t)$ is defined by:

$$\int_{h_1}^{h_2} g(h,t)dh = \frac{A(h_1,h_2)}{\mathscr{A}} \tag{3}$$

The distribution depends on time due to two phenomena. On the one hand, there is a thermodynamical (algebraic) growth because sea ice exchanges energy with the atmosphere and the ocean. On the other hand, numerous mechanical processes influence this growth like for instance rafting, ridging and the formation of open water. One can show that a steady state of the thckness distribution is properly described by the following 2nd order linear ODE:

$$g''(h) + \left(\frac{1}{H} - \frac{q}{h}\right)g'(h) + \frac{q}{h^2}\ g(h) = 0 \quad \text{with real } q > 0,\ H > 0 \tag{4}$$

(a) (1p) Justify that this equation is solvable by the Frobenius method before effectively solving it.

(b) (2p) Use the Frobenious method to find the indicial and recurrence relation for the coefficients of the series solution.

(c) (1p) For one of the solutions of the indicial equation, the recurrence relation simplifies. Compute the sum of the series in this case. Normalize your solution such that $\int_0^{+\infty} g(h)dh = 1$.