Properties of the Dirac distribution

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1 Definition of the Dirac distribution by an integral over $[0, +\infty[$

Let f be a test function ; e.g. it can be a function with a finite support[1] or a Schwartz' function[2]. The Dirac distribution is usually defined by an integral over $\mathbb{R}$:

$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx \triangleq f(0) \tag{1}$$

In fact, the integration can be restricted to integral with contains 0. More generally, the function is evaluated at the point where the Dirac peak is located :

$$\int_{x_1}^{x_3} \delta(x - x_2)\, dx = 1, \quad \text{with} \quad x_1 < x_2 < x_3 \tag{2}$$

We are going to show that the Dirac distribution can also be defined by an integral over $[0, +\infty[$.

$$\begin{aligned} f(0) &= \int_{-\infty}^{+\infty} f(x)\delta(x)dx = \int_{0}^{+\infty} f(x)\delta(x)dx + \int_{-\infty}^{0} f(x)\delta(x)dx = \int_{0}^{+\infty} \big\{f(x)\delta(x) + f(-x)\delta(-x)\big\}dx \\ &= \int_{0}^{+\infty} \big\{f(x) + f(-x)\big\}\delta(x)dx \equiv \begin{cases} 2\int_0^{+\infty} f(x)\delta(x)dx & \text{if } f \text{ is even} \\ 0 \quad \text{if } f \text{ is odd} \end{cases} \end{aligned}$$

NB : $\delta(-x) = \delta(x)$ is a only formal identity, because δ is not a function.

Any function can be decomposed in an even part and an odd part[3] : $f \equiv E + O$ with $E(-x) = E(x)$ and $O(-x) = -O(x)$. We have just shown that the odd part does not contribute to the definition (1), the reason being $O(0) = 0$. Then, one can actually define the Dirac distribution as follows :

$$\int_{0}^{+\infty} f(x)\delta(x)dx \triangleq \frac{f(0)}{2} \tag{5}$$

One immediately infers the following useful identities :

$$\int_0^1 \delta(x)\, dx = \frac{1}{2} \tag{6}$$

$$\int_0^a \delta(x - a)\, dx = \frac{1}{2} \tag{7}$$

$$\int_0^{+\infty} e^{ikx}\delta(x)dx = \frac{1}{2} \tag{8}$$

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[1] The support of a function is the part of its domain where it is non-zero.

[2] A smooth function whose all derivatives are rapidly decreasing.

[3]

$$E(x) \triangleq \frac{f(x) + f(-x)}{2} \tag{3}$$

$$O(x) \triangleq \frac{f(x) - f(-x)}{2} \tag{4}$$

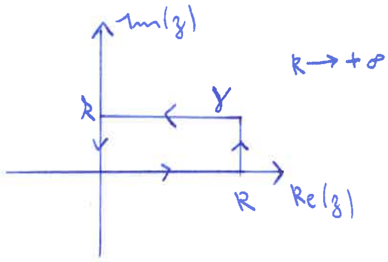

Figure 1: Contour γ

$x \mapsto e^{ikx}$ does not have a finite support and is not a Schwartz' function neither. However, equation (8) can be derived independently :

Proof. (8): We use a nice representation of the Dirac distribution :

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{\frac{-x^2}{2\epsilon}} \tag{9}$$

One can show that permutation of the limit with the integral is allowed. Then,

$$\int_0^{+\infty} e^{ikx}\delta(x)dx = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon}} \int_0^{+\infty} e^{\frac{-x^2}{2\epsilon}+ikx}dx = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon}}\, e^{-\frac{k^2\epsilon}{2}} \int_0^{+\infty} e^{\frac{1}{2\epsilon}(x+ik)^2}dx \tag{10}$$

Applying Cauchy integral theorem to the contour γ (see Figure 1), one gets :

$$\int_0^{+\infty} e^{\frac{1}{2\epsilon}(x+ik)^2}dx = \int_0^{+\infty} e^{\frac{x^2}{2\epsilon}}dx = \sqrt{2\epsilon}\,\frac{\sqrt{\pi}}{2} \tag{11}$$

In the end,

$$\int_0^{+\infty} e^{ikx}\delta(x)dx = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon}}\, e^{-\frac{k^2\epsilon}{2}}\, \sqrt{2\epsilon}\,\frac{\sqrt{\pi}}{2}dx = \frac{1}{2}\lim_{\epsilon \to 0} e^{-\frac{k^2\epsilon}{2}} \equiv \frac{1}{2} \tag{12}$$

□

2 A few identities

Here are a few more identities (g is a differentiable real-valued function) :

$$\delta\big(g(x)\big) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}, \quad \text{with} \quad g(x_i) = 0 \tag{13}$$

$$\int_{-\infty}^{+\infty} e^{ikx}dk = 2\pi\ \delta(x) \tag{14}$$

$$\int_0^{+\infty} \cos(kx)dk = \pi\ \delta(x) \tag{15}$$

$$\int_0^{+\infty} e^{ikx}dk = \pi\ \delta(x) \tag{16}$$

(13) and (14) are standard. They can be derived as follows :

Proof. (13) : For any test function f, the integral

$$\int_{-\infty}^{+\infty} \delta\big(g(x)\big)f(x)dx \tag{17}$$

is roughly the evaluation of f at points where g is equal to 0. For the sake of simplicity, let us assume that g has only one SIMPLE zero denoted x_0 and perform the change of variable

$$u - x_0 = g(x) \quad \Rightarrow \quad dx = \frac{du}{g'(x)} \tag{18}$$

which is such that when $u = x_0$, $g(x) = 0$ and so $x = x_0$. It suits perfectly with the essence of the Dirac delta. In addition, we will have to take the absolute value of the Jacobian $\frac{1}{g'(x)}$ and a priori write $x = g^{-1}(u)$ in order to get a proper integration with respect to the variable u. However, there is $\delta(u - x_0)$ in the integrand meaning that u can be replaced everywhere by x_0 or equivalently x can be replaced everywhere by x_0. In the end, this gives :

$$\int_{-\infty}^{+\infty} \delta\big(g(x)\big) f(x) dx = \int_{-\infty}^{+\infty} \delta(u - x_0) f(u) \, \frac{du}{|g'(x_0)|} \equiv \frac{f(x_0)}{|g'(x_0)|} \tag{19}$$

Note that $g'(x_0) \neq 0$ since x_0 is a SIMPLE of g. It appears that if g has a zero with a multiplicity, then $\delta\big(g(x)\big)$ does not make any sense.
When g has several zeros, one simply splits $\mathbb{R}$ into intervalles containing one zero each. □

Proof. (14) : Let us define the Fourier transformation and the inverse Fourier transformation for suitable functions or distributions g and h.

$$\mathcal{F}(g)(k) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) e^{-ikx} dx \tag{20}$$

$$\mathcal{F}^{-1}(h)(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(k) e^{ikx} dk \tag{21}$$

It is obvious that :

$$\mathcal{F}(\delta)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \tag{22}$$

One can show that the Dirac distribution belongs to a space for which $\mathcal{F}^{-1}$ is indeed the reciprocal transformation of $\mathcal{F}$. Therefore :

$$\delta = \Big(\mathcal{F}^{-1} \circ \mathcal{F}\Big)(\delta) \quad \Rightarrow \quad \delta(x) = \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\right)(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} dk \tag{23}$$

□

A simple way to get identity (15) is to notice that $\delta(x)$ is real. Then, take the real part of (14) and use the parity of cosine :

$$\int_{-\infty}^{+\infty} \cos(kx) dk = 2\pi \, \delta(x) \quad \Leftrightarrow \quad \int_{0}^{+\infty} \cos(kx) dk = \pi \, \delta(x) \tag{24}$$

Note that cosine is not integrable over $\mathbb{R}$, thus it is not surprising that its integration leads to a distribution instead of a function. Sine is not integrable neither over an infinite domain, nonetheless due to its parity it is clear that :

$$\int_{-\infty}^{+\infty} \sin(kx) dk = 0 \tag{25}$$

We eventually provide an independent derivation of (15) :

Proof. (15): We define for $\epsilon > 0$ the function δ_ϵ (see figure 2) as :

$$\delta_\epsilon(x) = \begin{cases} \frac{1}{\epsilon} & \text{if } |x| < \frac{\epsilon}{2} \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

One can show that $\lim\limits_{\epsilon \to 0} \delta_\epsilon = \delta$ moreover δ_ϵ is a function with a finite support so that :

$$\forall \epsilon > 0, \quad \delta_\epsilon = \Big(\mathcal{F}^{-1} \circ \mathcal{F}\Big)(\delta_\epsilon) \Rightarrow \quad \delta_\epsilon(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathcal{F}(\delta_\epsilon)(k) e^{ikx} dk \tag{27}$$

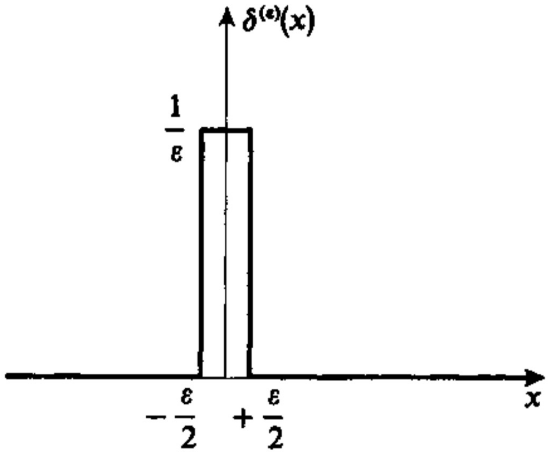

Figure 2: Function δ_ϵ

Hence, let us compute the Fourier transform of δ_ϵ :

$$\mathcal{F}(\delta_\epsilon)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta_\epsilon(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}\epsilon} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \frac{e^{\frac{ik\epsilon}{2}} - e^{-\frac{ik\epsilon}{2}}}{ik\epsilon} \equiv \frac{1}{\sqrt{2\pi}} \operatorname{sinc}\left(\frac{k\epsilon}{2}\right) \tag{28}$$

where we used Euler formula for sine and recognized sine cardinal $\operatorname{sinc}(x) \triangleq \frac{\sin(x)}{x}$. Plugging this result in (27) and taking the limit ϵ going to 0 gives :

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \operatorname{sinc}\left(\frac{k\epsilon}{2}\right) e^{ikx} dk = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \operatorname{sinc}\left(\frac{k\epsilon}{2}\right) \cos(kx) dk + i \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \operatorname{sinc}\left(\frac{k\epsilon}{2}\right) \sin(kx) dk$$

Sine cardinal is an even function so that its integral with sine is zero and its integral with cosine can be restricted to the range $[0, +\infty[$ with a factor 2. Thus,

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{0}^{+\infty} \operatorname{sinc}\left(\frac{k\epsilon}{2}\right) \cos(kx) dk = \frac{1}{\pi} \int_{0}^{+\infty} \lim_{\epsilon \to 0} \operatorname{sinc}\left(\frac{k\epsilon}{2}\right) \cos(kx) dk = \frac{1}{\pi} \int_{0}^{+\infty} \cos(kx) dk \tag{29}$$

Once again, one can show that permutation of the limit with the integral is allowed. We made use of the well-known limit $\lim_{x \to 0} \operatorname{sinc}(x) = 1$. □

Identity (16) turns out to be true even though its derivation is elusive. The reason for this is that, (15) being rigorously proved, it is equivalent to state the following :

$$\int_{-\infty}^{0} \sin(kx) dk = \int_{0}^{+\infty} \sin(kx) dk = 0 \tag{30}$$

It means that infinity is considered as an infinitely countable numeber of periods, no matter the size x of the period. One obtains zero because the integral vanishes over one period.

Some people actually consider (16) as a convention. Basically, it is always possible to use (14) or (15) instead.