Tutorial on Sturm-Liouville problems and Green's functions

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1 A general property of solutions of the 1D Schrödinger equation

The 1D Schrödinger equation for the wave function $\Psi(x,t)$ of a particle of mass m in a potential V is :

$$i\hbar\,\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\,\frac{\partial^2\Psi}{\partial x^2} + V(x)\;\Psi(x,t) \tag{1}$$

1) Separate variables to get the following 2nd order linear ODE :

$$-\,\psi''(x) + U(x)\;\psi(x) = \epsilon\;\psi(x) \tag{2}$$

using rescaled quantities U and ϵ such that $V = \frac{\hbar^2}{2m}\,U$ and $E = \frac{\hbar^2}{2m}\,\epsilon$. You don't need to solve the time equation.
A bound state is a state of energy ϵ such that $\epsilon < \lim\limits_{x\to-\infty} U(x)$ AND $\epsilon < \lim\limits_{x\to+\infty} U(x)$. It means that the particle is trapped in the potential and cannot escape.
A scattering state is a state of energy ϵ such that $\epsilon > \lim\limits_{x\to-\infty} U(x)$ OR $\epsilon > \lim\limits_{x\to+\infty} U(x)$. It means that the particle can go away.

2) Justify with a physical argument why for bound states $\lim\limits_{x\to-\infty}\psi(x) = 0$ AND $\lim\limits_{x\to+\infty}\psi(x) = 0$.

3) For a given energy ϵ, we look for the steady wave function $\psi(x)$. What kind of general problem is it ? The equation being self-adjoint, what kind of more specific problem is it ?

Technical remark: Differential operators are unbounded operators so that they can be Hermitian without being necessarily self-adjoint. Physicists usually do make not any distinction between Hermitian and self-adjoint ; the difference lies in the fact that an operator has often a smaller domain than its adjoint so that, even though they have the same action on this common domain, they are not the same. The issue is that the spectral theorem requires to have self-adjoint operators. One can show that the Hamiltonian is self-adjoint when it is defined over the Hilbert space of square integrable functions with a compact support. So it clearly works for a study of bound states. The case of scattering states is much more tricky.

4) We now consider only bound states. In this case, justify that $\epsilon \in \mathbb{R}$. Let ψ_1 and ψ_2 the solutions of equation (2) for given ϵ_1 and ϵ_2 respectively. For arbitrary $(a,b) \in \mathbb{R}^2$ show thanks to an integration by parts that :

$$\Big[W(\psi_1,\psi_2)\Big]_a^b = (\epsilon_1 - \epsilon_2)\;\int_a^b \psi_1(x)\psi_2(x)dx \tag{3}$$

where $W(\psi_1,\psi_2)$ is the Wronskian of ψ_1 and ψ_2.

5) Infer that in 1D any bound state has no degeneracy, meaning if two wave functions describe states of same energy then they are proportional.

2 On the importance of BCs in Sturm-Liouville problems

The Legendre differential equation is:

$$(1 - x^2)y''(x) - 2xy'(x) + l(l+1)y(x) = 0 \quad l \in \mathbb{N} \tag{4}$$

1) Write it in the self-adjoint form, using a weight factor if necessary.

2) Show that on the interval $[-1, 1]$ the corresponding linear operator is Hermitian, a priori irrespectively of the boundary conditions.

3) $P(x) = x$ and $Q(x) = \mathrm{argth}(x)$ are solutions of equation (4) corresponding to different eigenvalues. Show that $\int_{-1}^{1} P(x)Q(x)dx \neq 0$.

4) Explain the paradox.

3 Quantum harmonic oscillator

Take a potential $U(x) = \frac{x^2}{4}$ in the stationary Schrödinger equation (2). One obtains the so-called parabolic-cylinder equation parametrized by $\nu \in \mathbb{R}$ defined as $E = \nu + \frac{1}{2}$.

1) Perform the transformation $\psi(x) = e^{-\frac{x^2}{4}} H(x)$ to obtain the Hermite differential equation.

2) Show with the Frobenius method that the eigenvalues of this problem are $\nu \in \mathbb{N}$.

4 Green's function with a BC at infinity

Compute the Green's function of the following linear equidimensional ODE

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 4y(x) = f(x) \tag{5}$$

with the condition that $y(x)$ is finite $\forall x \in [0, +\infty[$.

5 Generation of waves by the wind: second round

Find back the solution of the following initial-value problem thanks to the method of Green's functions

$$\frac{d^2\eta}{dt^2} - 2\gamma\frac{d\eta}{dt} + \omega^2\eta(t) = p_a(t) \tag{6}$$

$$\eta(0) = 0 \quad \text{and} \quad \left.\frac{d\eta}{dt}\right|_{t=0} = 0 \tag{7}$$

for an arbitrary function p_a. Assume $\gamma \ll \omega$ as previously.