PH 6458/ 4258: Gravitation and Cosmology (Fall 2019)
Homework Set 1*

September 13, 2019

1. Starting from the expression for,

$$\vec{g}(\vec{x}) = -G_N \; d^3\vec{x}\rho(\vec{x'})\frac{(\vec{x} - \vec{x'})}{|\vec{x} - \vec{x'}|^3}$$

   show that,

$$\vec{\nabla} \cdot \vec{g}(\vec{x}) = -4\pi G_N \rho(\vec{x}).$$

2. **Prove the Uniqueness Theorem.** The theorem states, the solution, $\Phi(\mathbf{x})$ to Poisson equation in a region of space is unique if the potential is specified at the boundary of the region or if the normal derivative of the potential is specified at boundary of that region.

3. **Prove the Mean Value Theorem**. This theorem states that if $\Phi(\mathbf{x})$ satisfies the Laplace equation i.e. for regions outside mass sources, then the value of the potential at any point P (which can be taken to be the origin of coordinate system without any loss of generality), is given by the mean value of the potential over a sphere (of any radius) which is centered at P.

$$\Phi(\mathbf{x}_P) = \frac{1}{4\pi R^2} \; dS \; \Phi(\mathbf{x}'), \tag{1}$$

   where dS is the area element (magnitude) on the surface of a sphere of radius R. Note that the sphere is centered at P See equation (1).

4. Derive the third term in the multipole expansion of the Newton's scalar potential,

$$\Phi(\vec{x}) = -G_N\frac{M}{|\vec{x}|} - G_N\frac{x^i}{|\vec{x}|^3} \; d^3\vec{x'}x^i\rho(\vec{x'}) - \frac{1}{2!}G_N\frac{x^i x^j}{|\vec{x}|^5}Q^{ij} + \ldots$$

   where,

$$Q^{ij} = \; d^3\vec{x'}\rho(\vec{x'})(3x'^i x'^j - \delta^{ij}x^2),$$

*Due on Fri, Sept 20.

is the *quadrupole moment tensor* of the mass distribution. Hint: You will need to write down the Taylor expansion for $\frac{1}{|\vec{x}-\vec{x'}|}$, about $\vec{x'} = 0$, up to first three terms. The first two terms will give the monopole and dipole moments/terms as shown in the lecture, the third piece is the quadrupole moment contribution.

5. Show that the Quadrupole moment tensor of a spherical mass distribution of radius R vanishes. (Hint: Use isotropy/ rotational symmetry but not homogeneity).

6. **Expression of the tidal force in a spaceship**: Consider the situation where we have a spaceship orbiting the earth at a radius, x_0 (i.e. it is the distance between the center of the earth to the center of the spaceship). Now we have two frames, first is the frame fixed on earth with its origin of coordinates at the center of earth. We will assume this to be an inertial frame. The second frame is the frame attached to the spaceship and in which the origin of coordinates is located at the center of mass of the spaceship. This spaceship-frame being accelerated wrt earth-frame, is a non-inertial frame. Show at a point, P in the spaceship whose coordinates are given by $\mathbf{x}$ in the spaceship frame (and say $\mathbf{X}$ in the earth frame), experiences a force due to variation of the earth's gravitational field in the spaceship, given by:
$$f^i = x^j \frac{\partial F^i}{\partial x^j} = -x^j m \left(\frac{\partial^2 \Phi}{\partial x^j \partial x^i} \right) = -x^j mc^2 \, R^i{}_{0j0}.$$
We have introduced this weird looking notation, $R^i{}_{0,j0}$ because it will coincide with Riemann tensor which represents the curvature of the spacetime as a result of gravity.