Fall 2019: Gravitation and Cosmology (PH6458/ EP4258)

Lorentz covariant formulations of fluid dynamics and Electrodynamics*

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1 Principle of covariance in special relativity and Poincaré invariance of the action

In class I mentioned that laws of physics (or equivalently equations of motion of physical systems) must have the "same form" in all inertial frames. From a practical point of view this means, one should be able to write down physical law/equations in the form,

$$T^{\cdots}_{\cdots} = 0.$$

The quantity on the left hand side, namely $T^{\cdots}_{\cdots}$, is some tensor (could be any generic(p,q)-rank tensor). This guarantees that if one performs a Lorentz transformation (or more generally a Poincaré transformation), the equation in the new inertial frame will be identical looking, i.e.,

$$T'^{\cdots}_{\cdots} = 0!$$

This is ensured because tensors transform homogeneously under a Lorentz transformation,

$$T'^{\cdots}_{\cdots} = \Lambda \ldots \bar{\Lambda} \ldots T^{\cdots}_{\cdots},$$

where Λ and $\bar{\Lambda}$ are the direct and inverse Lorentz transformation matrix elements.

So the next logical question is how do we go about writing down physical equations/laws in tensor form to begin with. To answer this question we recall that in general we obtain equations of motion for any physical system are obtained from an action (functional), I by applying the variational principle,

$$\delta I = 0.$$

The principle of covariance is then tantamount to demanding that the action, I itself is invariant under Poincaré transformation. One might ask, why does the action has to be a scalar (i.e. invariant) and not a tensor of a more general kind. The answer I believe cannot be found in the classical realm but instead lies in the quantum real where the action determines probability amplitudes of various processes, e^{iI}. If the action was a tensor (instead of a scalar), then such probability amplitudes will change when going from one inertial frame to another inertial frame. Thus, quantum mechanics demands us the action to be a

*Based on lectures on Sept. 25, 27 & 30.

scalar under Lorentz (in fact Poincaré) transformations. **Thus, given a physical system our starting point should be an action which is a Lorentz scalar.** In addition we will demand the action to also satisfy some extra physical requirements. We list all of them them here

1. The action must be a scalar under Lorentz transformations and translations. (This is best understood when one realizes that in quantum theory the action for a given trajectory represents the probability amplitude, e^{iI} for that trajectory. Different inertial frames must compute the same probabilities for the same trajectory, hence one must have, $e^{iI} = e^{iI'}$. This will automatically hold if $I = I'$).
2. The action must be real. This has to do with probability conservation (unitarity) in quantum mechanics, complex values of action let to loss of unitarity. The amplitude e^{iI} will have a norm different than unity if $I \in \mathbb{C}$.
3. The action must involve at best two derivatives. Higher derivatives will lead to violation of causality or create instability (Ostrogradsky's theorem)
4. In case of field theories, the Lagrangian density should be local, $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$.
5. The action might be a scalar/invariant under other transformations (internal symmetries)

We will see that these conditions will be sufficient to construct action functional for physical theories, especially field theories of fundamental interactions in nature.

2 The Free Point Particle

The action of a point particle is,

$$I_{pp}[x^\mu(\tau)] = -mc \int \sqrt{-ds^2} \tag{1}$$

where ds^2 is the Lorentz invariant squared interval:

$$ds^2 = dx^\mu dx_\mu = \eta_{\mu\nu} dx^\mu dx^\nu = d\mathbf{x} \cdot d\mathbf{x} - dt^2.$$

Since all three quantities, the mass, m, the signal speed, c and the ds^2, are Lorentz invariant, the action being the product of three Lorentz invariant quantities is itself a Lorentz invariant. The equations of motion are obtained by the principle of stationary action or least action - vary the action and set the variation to zero, $\delta I_{pp} = 0$. Since the point particle action is proportional to the four dimensional (space-time) "path length", $\sqrt{-ds^2}$, the principle of least (or stationary) action for the point particle is actually the "principle of shortest(stationary) path or distance".

Now a one-dimensional curve is usually defined by a single parameter, i.e. the coordinates of points on the worldline are functions of a single parameter[1]. So we introduce a "worldline parameter", λ which labels the worldline of the point particle i.e. all points on the world line are now functions of λ,

$$x^\mu = x^\mu(\lambda).$$

[1]Analogously a surface or sheet is two-dimensional object, meaning the points on the surface are functions of two independent parameters, $x^\mu(\lambda_1, \lambda_2)$.

Then we can rewrite the squared interval in parametric form as well:

$$ds^2 = dx^\mu(\lambda)\, dx_\mu(\lambda) = d\lambda^2 \,\frac{dx^\mu(\lambda)}{d\lambda}\,\frac{dx_\mu(\lambda)}{d\lambda}. \tag{2}$$

So now the action looks like

$$I_{pp}[x^\mu(\tau)] = -mc \int d\lambda \,\sqrt{-\dot{x}^\mu(\lambda)\dot{x_\mu}(\lambda)}. \tag{3}$$

where the overdot means total derivative wrt the worldline parameter, $\lambda : \dot{x}(\lambda) \equiv \frac{dx(\lambda)}{d\lambda}$.

As we shall shortly learn that reparameterization symmetry of the point particle action gives us the freedom to choose anything as a worldline parameter, let's use that to set the worldline parameter to be same as the lab frame time, t[2] i.e.

$$x^0(t) = t, \mathbf{x}(t).$$

We are using natural units, whereby $c = 1$. The parametric derivatives are then (lab frame) time derivatives, $\frac{dx^\mu}{dt}$. Now $\frac{dx^0}{dt} = 1$ while the time-derivative of the spatial coordinates are,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(t),$$

i.e. the lab-frame velocity. Then, the parametric form of the squared interval (2) looks like,

$$ds^2 = dt^2 \frac{dx^\mu(t)}{dt}\,\frac{dx_\mu(t)}{dt}.$$

The μ-index is contracted i.e. summed over all $\mu = 0, 1, 2, 3$ and when we write out the sum explicitly, we get,

$$ds^2 = dt^2 \frac{dx^0(t)}{dt}\,\frac{dx_0(t)}{dt} + \frac{dx^i(t)}{dt}\,\frac{dx_i(t)}{dt}$$

where the index i runs over the spatial dimensions, $i = 1, 2, 3$. The student is invited to plug the above formula in the action, (3) and realizing that, $\frac{dx^i}{dt} = v^i$, in the following exercise as a homework.

2.1 Equation of motion for the free relativistic point particle

Let's vary the action to get the equations of motion of the point particle. We can do this in two different ways. The first one I will use the Euler-Lagrange equations applied to the point particle Lagrangian thinking of the world line parameter, λ as time. Then,

$$L\left(x^\mu, \dot{x}^\mu, \lambda\right) = -m\sqrt{-\dot{x}^\mu \dot{x}_\mu}$$

[2] Later we will opt for a different and more convenient form of reparametrizations called the proper time parametrization in which we choose the time in the frame attached to the point particle (say τ) itself as a parameter,

$$\lambda = \tau.$$

Since in the frame attached to the particle the particle does not move at all, $d\mathbf{x}' = 0$. So the invariant squared interval is,

$$ds^2 = d\mathbf{x}'^2 - d\tau^2 = -d\tau^2.$$

where the $\{x^\mu\}$ are the generalized coordinates and the $\left\{\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}\right\}$ are the generalized velocities. The equation of motion for the coordinate x^μ is

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \frac{\partial L}{\partial x^\mu}.$$

Plugging in $\frac{\partial L}{\partial x^\mu} = 0$, and the canonical momenta, $P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\,\dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}}$, we get the equation of motion to be,

$$\frac{d}{d\lambda}\left(\frac{\dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}}\right) = 0. \tag{4}$$

We can make things a little bit nicer in appearance by introducing a relativistic definition of velocity, "4-velocity" defined by

$$u_\mu \equiv \frac{\dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}}. \tag{5}$$

In terms of this the equation of motion of the point particle is

$$\frac{du_\mu}{d\lambda} = 0. \tag{6}$$

This is the equation of motion for the point particle for *arbitrary* paremeterization. So to get some insight we can ask how does it look in the lab frame where we can take the lab frame time, t to be the worldline parameter, i.e. $\lambda = t$. In that case the proper time velocity are,

$$u^\mu = \frac{c\dot{x}^\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} = \left(\frac{c^2}{\sqrt{c^2 - \mathbf{v}^2}}, \frac{\mathbf{v}c}{\sqrt{c^2 - \mathbf{v}^2}}\right) = (\gamma c, \gamma \mathbf{v})\,.$$

Hence we have,

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1 - \frac{\mathbf{v^2}}{c^2}}}\right) = \frac{d}{dt}\left(\frac{\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v^2}}{c^2}}}\right) = 0.$$

This immediately gives,

$$\frac{d\mathbf{v}}{dt} = 0,$$

as predicted by Newton's first law.

2.2 Proper time reparametrizations : Straight lines trajectories

We can make things even more nicer in appearance if we choose a new parameter to relabel the world line. Say, we choose a new parameter. $\tau(\lambda)$ defined by integrating the following equation

$$\frac{d\tau(\lambda)}{d\lambda} = \sqrt{-\dot{x}^\mu \dot{x}_\mu}$$

or,

$$d\tau = d\lambda\sqrt{-\dot{x}^\mu \dot{x}_\mu}.$$

So in terms of τ we get the equation of motion to be,

$$\frac{d^2 x^\mu}{d\tau^2} = 0. \tag{7}$$

This new parameter, τ is in fact special and is equal to the proper time of the point particle (time in the rest frame of the particle), because the square of it is

$$d\tau^2 = (d\lambda)^2 \left(-\frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda} \right) = -dx^\mu dx_\mu!$$

i.e. the squared interval. It is called proper time because, it has the same sign as t on the rhs,

$$d\tau^2 = -dx^\mu dx_\mu = dt^2 - d\mathbf{x} \cdot d\mathbf{x}.$$

Solutions to this equation is of course straight lines,

$$x^\mu(\tau) = v^\mu\, \tau + w^\nu.$$

where, v, w are constants of integration.

One can check that the same equation (7) also holds when the parameter is the "proper length", $s(\lambda)$ defined by square root of the squared spacetime interval,

$$ds = \sqrt{ds^2} = \sqrt{\dot{x}^\mu \dot{x}_\mu} = \sqrt{d\mathbf{x} \cdot d\mathbf{x} - dt^2}.$$

Again the reason why we call it proper length is because the ds^2 and $d\mathbf{x}^2$ have same sign.

2.3 Symmetries of the point particle action

A great virtue of the action formulation is that it reflects the symmetries of the system manifestly. For the point particle, we have

- Since the mass, m, τ are Lorentz invariants, and all indices are summed over inside the square root, the action is manifestly Lorentz invariant i.e. under $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$.
- The action is in addition Poincare invariant. This is because the action only contains derivative of x, the derivative kills any constant shift, a

$$\frac{d(x+a)}{d\lambda} = \frac{dx}{d\lambda}$$

- The action is "Reparameterization invariant" i.e. changing the parameter which is labeling the world line, $\lambda \to \lambda' = f(\lambda)$. This symmetry is distinguished from the above two by the fact that the symmetry transformation, $f(\lambda)$ is a function of of the worldline parameter, λ i.e. it varies along the length of the worldline. This is an example of a *gauge* symmetry[3].

2.4 Conserved Charges from the Action

Another virtue of the action is that one can easily deduce the conserved charges such as linear and angular momentum, energy, etc. very easily. This process of extracting conserved charges from the action is called the **Noether procedure**. This is because the procedure is based on Noether's theorem, which

[3]Earlier $\Lambda^\mu{}_\nu$ and the shift a we constants, i.e. same all along the worldline. Such symmetries are called *global* symmetries.

is a very deep and fundamental result in physics, applicable in both classical and quantum system and which states:

Noether's Theorem: Corresponding to every continuous global symmetry of a physical system, there exists a conserved charge. If the symmetry is local (gauge), then there is no associated conserved charge. (Gauge symmetries represent redundancies of description)

From Sec. (2.3), we state the three conserved charges of the point particle:

Symmetries of the action	Conserved Charges
Lorentz symmetry (boosts, rotations), $\Lambda^{\mu}{}_{\nu}$	$M^{\mu\nu} = m\ (\dot{x}^{\mu}x^{\nu} - \dot{x}^{\nu}x^{\mu})\,/\sqrt{-\dot{x}^2}$
Translation symmetry, a^{μ}	$p^{\mu} = m\,\dot{x}^{\mu}/\sqrt{-\dot{x}^2}$
Reparameterization symmetry, $\tau \to f(\tau)$	$\times$

The important quantity here is the linear momentum 4-vector,

$$p^{\mu} = m\dot{x}^{\mu}/\sqrt{-\dot{x}^2} = m\,u^{\mu} = (\gamma m, \gamma m\mathbf{v})\,.$$

The zeroth component, $p^0 = \gamma m$ is the relativistic energy, and in the rest frame it is just the invariant mass, $p_0^0 = m$.

2.5 Motion of point particle under influence of forces

This requires us to make a 4-vector version of Newton's second law,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt}.$$

One can easily guess the formal generalization of the second law,

$$F^{\mu} = \frac{dp^{\mu}}{dt} = m\frac{du^{\mu}}{d\tau}.$$

The quantity on the lhs is the 4-force which is the relativistic counterpart of a 3-vector force in non-relativistic mechanics. However the zeroth component of this force 4-vector is the rate of change of energy, i.e. *power*. An example of the 4-force vector is the Electromagnetic force on a charged particle,

$$F^{\mu} = qF^{\mu\nu}u_{\nu}.$$

One can check that in the non-relativistic limit it reduces to the familiar Lorentz force law,

$$\mathbf{F} = q\ (\mathbf{E} + \mathbf{v}\times\mathbf{B})\,.$$

The details can be found in Sec. 5.4.

3 Continuum limit: Tensor fields

So far we have looked at a single 4-variable, e.g. position of single particle. This can be generalized to multiple but still finite particles with an index labeling the particle, $x_i(t)$. However consider the case when we keep on adding new degrees of freedom so that at the end we have an infinite number of variables, one at each point in space. Such a collection of infinite number of variables at each point in space is called a **field**. For example, consider any point on a string strung along the horizontal direction, say along the x-axis. When the string is oscillating (transverse), at any instant of time, t each point on the string will have a vertical displacement in the y-direction. We can denote the vertical displacement by a field where we specify the location/point on the string, x and the time when it occurs, say t by a **displacement field**

$$y(x,t).$$

Similarly for longitudinal waves through a gas, at each point in the gas, we have a density fluctuation field,

$$\Delta\rho(\mathbf{x},t)$$

which is the change in density of the material from its normal equilibrium density.

Generalizing, we can define a "tensor fields" which means at each point in space we have a tensor-valued quantity which is dependent on time. For example, a scalar field, $\phi(x^\mu)$ or a vector field, $V^\mu(x)$. Then it is natural to ask what is the transformation law for such tensor fields under a Lorentz transformation. The answer becomes clear after a little thought,

$$\begin{aligned} x \to x' &= \Lambda x \\ \phi(x) \to \phi'(x') &= \phi(x), \\ V^\mu(x) \to V'^\mu(x') &= \Lambda^\mu{}_\nu\, V^\nu(x), \\ \omega_\mu(x) \to \omega'_\mu(x) &= \Lambda_\mu{}^\nu \omega_\nu(x). \end{aligned}$$

This transformation is often called the **passive transformation.**

Often, we shall be interested in the **active transformation** of a tensor field after an LT, defined by,

$$\begin{aligned} \Delta V^\mu(x) &= V'^\mu(x) - V^\mu(x). \\ &= \Lambda^\mu{}_\nu\, V^\nu(\Lambda^{-1}x) - V^\mu(x) \\ &= \Lambda^\mu{}_\nu\, V^\nu \left(\Lambda_\beta{}^\alpha\, x^\beta\right) - V^\mu(x) \\ &= \left(\delta^\mu_\nu + \omega^\mu{}_\nu\right)\, V^\nu \left(x^\alpha + \omega_\beta{}^\alpha\, x^\beta\right) - V^\mu(x^\alpha) \\ &= \omega^\mu{}_\nu\, V^\nu(x) + \omega_\beta{}^\alpha\, x^\beta\, \partial_\alpha V^\mu(x^\alpha) \\ &= \omega^\mu{}_\nu\, V^\nu(x) + \frac{1}{2}\omega^{\alpha\beta}\, \mathcal{M}_{\alpha\beta} V^\mu, \qquad \mathcal{M}_{\alpha\beta} = [x_\alpha\, \partial_\beta - x_\beta\, \partial_\alpha] \end{aligned}$$

Observe that $\mathcal{M}$ is similar form as the angular momentum operator in quantum mechanics (except the i's are missing).

4 Covariant Fluids

The fluid field is parameterized by the density, $\rho(\mathbf{x},t)$ and the current $\boldsymbol{j}(\mathbf{x},t) = \rho(\mathbf{x},t)\,\boldsymbol{v}(\mathbf{x},t)$, where $\boldsymbol{v}(\mathbf{x},t)$ is the fluid velocity at $\mathbf{x}$ and at time t. Now, a well known equation for the fluid flow which represents conservation of mass is the continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0.$$

In the context of a fluid we usually talk about two frames, again the Lab frame,S and the fluid frame, S' which is attached to a fluid element. Let's choose the direction of velocity of the fluid element to be the common X-axis of both frames. Now let's look at the expression for the density. In the lab frame this is,

$$\rho = \frac{\Delta m}{dx\,dy\,dz}.$$

In the fluid frame, S', this is,

$$\rho' = \frac{\Delta m}{dx'\,dy'\,dz'}.$$

Since in the Lab frame the moving fluid element will appear to be Lorentz contracted along the x-direction,

$$dx = \frac{dx'}{\gamma},$$

while transverse length/dimensions of the fluid element are unchanged,

$$dy = dy', \quad dz = dz'.$$

This implies one can write

$$\rho = \gamma\rho',\; j_x = \beta c \rho = \gamma\beta(c\rho'),$$

Since the velocity of the fluid in the fluid rest frame is zero, we have, $j'_x = 0$. So we can write down an equation,

$$\rho = \gamma\rho' + \gamma\frac{\beta}{c}j'_x,\; j_x = \gamma\beta(c\rho') + \gamma j'_x$$

$$\begin{aligned}
\begin{pmatrix} c\,\rho \\ j_x \end{pmatrix} &= \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} c\,\rho' \\ j'_x \end{pmatrix}, \\
\begin{pmatrix} c\,\rho' \\ j'_x \end{pmatrix} &= \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} c\,\rho' \\ j'_x \end{pmatrix}
\end{aligned}$$

This equation is identical to the Lorentz transformation for the position 4-vector. From this one can deduce the existence of a new four vector for the fluid, the "4-current",

$$j^\mu = \left(j^0, \boldsymbol{j}\right), j^0 = c\rho.$$

The equation of continuity can then be written in a frame invariant form,

$$\partial_\mu j^\mu = 0$$

with equal number of up and down indices.

4.1 Particle number density 4-vector

The conservation of mass equation can also be converted into a conservation of particle number. To this end we note that, one can define a particle number current density, which is a 4-vector by,

$$N^\mu = n_0\, u^\mu$$

where, n_0 is the particle number density in the fluid rest frame and u^μ is the 4-velocity of the fluid center of mass. The mass-density 4-vector then can be expressed as, $j^\mu = m\, N^\mu$, where m is the mass of each fluid particle. The continuity equation for the particle number current density is then,

$$\partial_\mu N^\mu = 0.$$

Writing this out explicitly,

$$\begin{aligned} 0 &= \partial_0 N^0 + \partial_i N^i, \\ &= \frac{\partial\left(n\, u^0\right)}{\partial t} + \boldsymbol{\nabla}\cdot\left(n\, \mathbf{u}\right) \end{aligned}$$

Now by definition,

$$u^\mu = \left(\gamma, \gamma\mathbf{v}\right), \quad \gamma = \left(1-\mathbf{v}^2\right)^{-\frac{1}{2}},$$

and we have the continuity equation,

$$\frac{\partial}{\partial t}\left(\frac{n}{\sqrt{1-\mathbf{v}^2}}\right) + \boldsymbol{\nabla}\cdot\left(\frac{n\mathbf{v}}{\sqrt{1-\mathbf{v}^2}}\right) = 0. \tag{8}$$

Note that n, the rest frame particle number density can vary with time and position, so it cannot be taken out the derivatives.

4.2 Fluid Stress tensor and the form for Ideal fluids

There is one more conservation law which holds for the fluids, which is that of energy-momentum. Energy momentum density is described by a two-index tensor, dubbed as the energy-momentum-stress tensor or the stress tensor for short. It is denoted by $T^{\mu\nu}$ and it obeys the continuity equation,

$$\partial_\mu T^{\mu\nu} = 0.$$

The various components of the stress tensor represent the following flows of physical quantities,

- T^{00} = Energy density
- T^{i0} =Density of i-th component of Momentum being transported by the fluid
- T^{0i} =Energy Flux density (i.e. Energy flowing out per unit time per unit area normal to the i-th direction)
- T^{ij} = ij- Momentum Flux density (i.e. the flow of i-th momentum component in the j-direction per unit time, per unit normal area)

In particular, for a fluid made up of non-interacting particles (*dust*), one can easily check that,

$$T^{\mu\nu} = n_0 m \, u^\mu u^\nu \tag{9}$$

where n, as before, is the rest frame particle number density and u^μ is the fluid 4-velocity (CM). Here one notes that for *dust*, the stress tensor is symmetric,

$$T^{i0} = T^{0i}, \quad T^{ij} = T^{ji}$$

but that might not be so in general.

An *ideal fluid* is defined by the following form of the stress tensor in the rest frame (comoving frame),

$$\begin{aligned} T_0^{00} &= \rho, \\ T_0^{i0} &= T_0^{0i} = 0, \\ T_0^{ij} &= P\,\delta^{ij}. \end{aligned}$$

The subscript 0 in $T_0^{\mu\nu}$ implies that it is a rest frame quantity, which is why there is no net momentum transported by the fluid and the quantities in the second line vanish. The third line represents two physical facts, first is that the ideal fluids cannot withstand shearing stress, i.e.

$$T_0^{ij} = 0$$

if $i \neq j$, and second that the diagonal components are identical because the fluid looks isotropic in the CM frame, namely,

$$T_0^{11} = T_0^{22} = T_0^{33} = P.$$

.In matrix form, one has,

$$T_0^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}.$$

Since the ideal fluid stress tensor is symmetric in the rest frame, it will be so in *any* frame. The form of the ideal fluid stress tensor in a general frame where the fluid is moving with a velocity $\mathbf{v}$, can be shown to be given by the expression,

$$T^{\mu\nu} = (\rho + P)\ u^\mu\, u^\nu + P\,\eta^{\mu\nu} \tag{10}$$

where as before, $u^\mu = (\gamma, \gamma\mathbf{v})$, $\gamma = \left(1 - \mathbf{v}^2\right)^{-1/2}$. Now notice that if we choose $P = 0$, the ideal fluid stress tensor form (10) reduces to the dust form, (9). This is why we call dust *pressure free* fluid.

Next we write down the continuity equation representing conservation of energy-momentum for the ideal fluid,

$$\partial_\mu T^{\mu\nu} = 0.$$

Substituting the ideal fluid stress tensor form (10),

$$\begin{aligned} 0 &= \partial_\mu \left((\rho + P)\ u^\mu\, u^\nu + P\, \eta^{\mu\nu} \right) \\ &= \partial_\mu \left((\rho + P)\ u^\mu\, u^\nu \right) + \partial^\nu P. \end{aligned}$$

Again one cannot assume that the rest frame pressure and density are constant over time and position in the fluid, they are fields i.e. which are functions of both position (location) and time. Taking $\nu = 0$ we get,

$$\begin{aligned} 0 &= \partial_\mu \left((\rho + P)\ u^\mu\, u^0 \right) + \partial^0 P \\ &= \frac{\partial}{\partial t} \left((\rho + P)\ u^0\, u^0 \right) + \partial_i \left((\rho + P)\ u^i\, u^0 \right) - \frac{\partial P}{\partial t} \\ &= \frac{\partial}{\partial t} \left(\gamma^2 \left(\rho + P \right) \right) + \partial_i \left(\gamma^2 v^i \left(\rho + P \right) \right) - \frac{\partial P}{\partial t}. \end{aligned} \tag{11}$$

Similarly, setting the free index, $\nu = i$, we get,

$$\begin{aligned} 0 &= \partial_\mu \left((\rho + P)\ u^\mu\, u^i \right) + \partial^i P \\ &= \frac{\partial}{\partial t} \left((\rho + P)\ u^0\, u^i \right) + \partial_j \left((\rho + P)\ u^j\, u^i \right) + \partial_i P \\ &= \frac{\partial}{\partial t} \left(\gamma^2 v^i \left(\rho + P \right) \right) + \partial_j \left(\gamma^2 v^i v^j \left(\rho + P \right) \right) + \partial_i P. \end{aligned} \tag{12}$$

The above two equations can be combined into a single vector equation, which is the relativistic version of Euler equation of fluid dynamics, *viz.*,

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v} \cdot \boldsymbol{\nabla} \right) \mathbf{v} = - \left(\frac{1 - \mathbf{v}^2}{P + \rho} \right) \left[\boldsymbol{\nabla} P + \mathbf{v} \frac{\partial P}{\partial t} \right]. \tag{13}$$

5 Classical Electrodynamics in Lorentz covariant form

5.1 Flow of point charges: Electric current density 4-vector

For a single point charge moving in a trajectory, $\mathbf{x}(t)$, we have the charge density,

$$\rho\left(t, \mathbf{x}\right) = q\, \delta^3 \left(\mathbf{x} - \mathbf{x}(t) \right), \tag{14}$$

while the current density is,

$$\mathbf{j}(t, \mathbf{x}) = \rho(t, \mathbf{x})\, \mathbf{v}(t) = q\, \mathbf{v}(t)\, \delta^3 \left(\mathbf{x} - \mathbf{x}(t) \right) \tag{15}$$

Of course both these expressions look non-covariant and we need to recast it in a form where the Lorentz transformation is manifest i.e. as the components of a four vector, j^μ. One could simply replace every quantity in expressions (14, 15) by their respective four-vector counterpart, leading to the tentative expression,

$$j^\mu(x^\nu) \stackrel{?}{=} q\, u^\mu(\tau)\, \delta^4\left(x^\nu - x^\nu(\tau)\right), \tag{16}$$

where τ is some invariant parameter finally to be identified with the proper time. However note that we have one extra Dirac delta compared to the non-relativistic expression and instead of lab frame time, t the parameter is τ. So this cannot be the correct answer. Let's try to guess the answer by switching back to t as a worldline parameter, then by comparing the rhs for $\mu = i$ i.e. by comparing to the rhs of (15). We have the rhs of the tentative expression (16),

$$\begin{aligned}
\text{RHS} &= q\, \frac{dx^i(\tau)}{d\tau}\, \delta(t - t(\tau))\, \delta^3\left(\mathbf{x} - \mathbf{x}(\tau)\right) \\
&= q\, \frac{dx^i(\tau(t))}{dt}\, \frac{dt}{d\tau}\, \delta(t - t(\tau))\, \delta^3\left(\mathbf{x} - \mathbf{x}(\tau)\right) \\
&= q\, \frac{dx^i(\tau(t))}{dt}\, \delta(\tau - \tau(t)\, \delta^3(\mathbf{x} - \mathbf{x}(\tau(t))) \\
&= q\, \frac{dx^i(t)}{dt}\, \delta(\tau - \tau(t))\, \delta^3(\mathbf{x} - \mathbf{x}(t)).
\end{aligned}$$

So now we can identify what is missing and what to do in order for the RHS to be same as the RHS of (15). We just need to integrate over τ! So we have the expression for the current density four-vector for a point electric charge to be,

$$j^\mu(x) = q \int d\tau\, u^\mu(\tau)\, \delta^4\left(x^\nu - x^\nu(\tau)\right). \tag{17}$$

5.2 Maxwell equations in Lorentz covariant form

The source of electric and magnetic fields is the charge density and current density. We already know how to fuse them into a single relativistic entity, the 4-current,

$$j^\mu = (c\rho, \boldsymbol{j}).$$

The rest is to figure out how do **E, B** combine into some relativistic format. The Maxwell's equation in terms of $\mathbf{E}, \mathbf{B}$ (in ***Heaviside-Lorentz*** units)

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} = \rho, &\qquad \boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j}, \\
\boldsymbol{\nabla} \cdot \mathbf{B} = 0, &\qquad \boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0.
\end{aligned}$$

The first line of two equations have the sources ρ and j on their RHS. The second line contains two equations which are "sourceless". So one thing is clear, there is a mismatch of number of degrees of freedom on the LHS (**E** and **B** have $3 + 3 = 6$ components) and RHS (source j^μ have 4 components). So we are missing something right now and a Lorentz covariant form is not obvious. To make further progress, we will need to recall our old friends, the scalar and vector potentials,

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}, \\
\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A}.
\end{aligned} \tag{18}$$

Now it is pretty obvious that one can combine the scalar and vector potential into a single 4-vector potential, usually called the **Maxwell Gauge field**,

$$A^\mu = (\Phi, \mathbf{A}) .$$

Now looking at (18) seems to indicate now needs to act with the derivatives on the gauge field to get E and B. Can we try something like $\mathbf{E}, \mathbf{B} \sim \partial_\mu A_\nu$? The answer is no because the number of components on both sides don't match up. $\mathbf{E}$ and $\mathbf{B}$ have a total of 6 components while $\partial_\mu A_\nu$ has $4 \times 4 = 16$ components (the indicesμ and ν can take 4 values each namely 0,1,2,4 , so the total number of index combination is 4×4). Clearly we have to select only a 6 component subset of these 16 derivatives but at the same time it has to be an "irrep" of Lorentz group. One possibility is to make a $(0, 2)$ type tensor, $F_{\mu\nu}$ defined by,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The $F_{\mu\nu}$ is called the (Maxwell) **field strength tensor**. This being manifestly antisymmetric in the two indices has ${}^4C_2 = 6$ components as well! Indeed one can check that,

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = \frac{\partial A^i}{\partial t} + \frac{\partial \Phi}{\partial x^i} = -E_i,$$

The magnetic field, B is contained in the components, F_{ij},

$$F_{ij} = \partial_i A_j - \partial_j A_i = \partial_i A^j - \partial_j A^i = \epsilon^{ijk} B^k.$$

Now the first Maxwell equation with sources can be expressed in a covariant way in terms of the field strength, $F_{\mu\nu}$ and current density 4-vector, j_μ as

$$\partial^\mu F_{\mu\nu} = -j_\nu. \tag{19}$$

The second Maxwell equation i.e. the one without sources/homogeneous equations can be expressed as,

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{20}$$

Since this equation has no source, we shall not call it an equation of motion. In fact, this equation is a consistency condition for field strength, it is called a **Bianchi identity.**

If we define a **dual field strength** tensor,

$$G_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \tag{21}$$

then the homogeneous Maxwell's equation can be expressed as,

$$\partial^\mu G_{\mu\nu} = 0. \tag{22}$$

5.3 Action for Electrodynamics

The action again must be made out Lorentz invariant quantities, like for example, $A_\mu A^\mu$, $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}G^{\mu\nu}$, $G_{\mu\nu}G^{\mu\nu}$, $j^\mu A_\mu$etc. From this list we will need just two terms,

$$I[A] = \int d^4x \, \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + j_\mu A^\mu \right) . \tag{23}$$

The curious factor of $-\frac{1}{4}$ is so that when we expand out the $F_{\mu\nu}F^{\mu\nu}$ term, we get a positive kinetic term with the correct coefficient for the potential, to wit, $\frac{1}{2}\dot{\mathbf{A}} \cdot \dot{\mathbf{A}}$. Similarly the second term in Lagrangian which is supposed to be $-V$, i.e. negative of potential energy is indeed so, since $j_\mu A^\mu = -\rho\,\Phi + ...$, which is the negative of the electrostatic potential energy.

5.4 Charged point particle in an Electromagnetic field

Let consider a point particle of mass, m and electric charge, q (a point electric monopole) in a background electromagnetic field, $F_{\mu\nu}$. We know that in the non-relativistic/Newtonian limit, we should get an equation of motion,

$$m\,\frac{d\mathbf{v}}{dt} = q\,\mathbf{E} + q\,\left(\mathbf{v} \times \mathbf{B}\right) . \tag{24}$$

We also know from the (neutral) point particle case in Sec. 2, the lhs of the above equation can be made covariant (i.e. a four vector/tensor) as follows,

$$m\,\frac{d\mathbf{v}}{dt} \to m\,\frac{du^\mu}{d\tau} .$$

The rhs is not that obvious since it has a two index object, $q\,F^{\mu\nu}$. So to make it match up with the lhs in terms of indices, we need to contract $F^{\mu\nu}$ with some (till now undetermined) vector, t_μ.If we look at the rhs of the noncovariant law, (24) we see the presence of a velocity and we are lead to suspect that this vector could be $u_\mu(\tau)$, the four-velocity vector. Indeed that is the correct guess. So now we can contract $F^{\mu\nu}$ on the rhs with $u_\nu(\tau)$ and we get the fully covariant version of the Lorentz force law,

$$m\,\frac{du^\nu(\tau)}{d\tau} = q\,F^{\mu\nu}(x(\tau))\,u_\nu(\tau). \tag{25}$$

What about the action, I_{pp}? How does it change from the *free* case? The answer can again be arrived at from a symmetry perspective. We can write down some terms which are Lorentz invariant *and* gauge invariant,

$$I_{EMp} = q \int d\tau\; F^{\mu\nu}\,u_\mu\,u_\nu .$$

But this vanishes, being a product of antisymmetric and symmetric tensors, so we might try,

$$I_{EMp} = q \int d\tau\; u_\mu\,A^\mu .$$

It can also be deduced by plugging in the point charge current density 4-vector form (17) in the interaction term $\int d^4x\, j_\mu A^\mu$ in the Maxwell action, (23).