PH 6458/ 4258: Gravitation and Cosmology (Fall 2019) Homework Set 1*

September 30, 2019

1. Starting from the expression for,

$$\vec{g}(\vec{x}) = -G_N \int d^3\vec{x}\rho(\vec{x'})\frac{(\vec{x} - \vec{x'})}{|\vec{x} - \vec{x'}|^3}$$

show that,

$$\vec{\nabla} \cdot \vec{g}(\vec{x}) = -4\pi G_N \rho(\vec{x}).$$

Solution: Taking divergence of both sides of $\vec{g}(\vec{x}) = -G_N \int d^3\vec{x}\rho(\vec{x'})\frac{(\vec{x}-\vec{x'})}{|\vec{x}-\vec{x'}|^3}$, and then using the well-known vector identity,

$$\boldsymbol{\nabla} \cdot \frac{\mathbf{x}}{|\mathbf{x}|^3} = 4\pi\delta^3(\mathbf{x}),$$

in the rhs, the result is obvious.

2. **Prove the Uniqueness Theorem.** The theorem states, the solution, $\Phi(\mathbf{x})$ to Poisson equation in a region of space is unique if the potential is specified at the boundary of the region or if the normal derivative of the potential is specified at boundary of that region.

Solution:
Let's assume that the solution to this in a region of space is non-unique and that there are two distinct solutions, Φ_1 and Φ_2 which satisfy the same boundary conditions at the boundary of this region,

$$\Phi_1|_{boundary} = \Phi_2|_{boundary},$$

$$\mathbf{n} \cdot \boldsymbol{\nabla}\Phi_1|_{boundary} = \mathbf{n} \cdot \boldsymbol{\nabla}\Phi_2|_{boundary}.$$

Now consider cooking up a new field, $\Phi \equiv \Phi_1 - \Phi_2$. This field of course satisfies the Laplace equation instead of the Poisson equation,

$$\begin{aligned} \boldsymbol{\nabla}^2\Phi &= \boldsymbol{\nabla}^2\Phi_1 - \boldsymbol{\nabla}^2\Phi_2 \\ &= 4\pi G_N\rho - 4\pi G_N\rho \\ &= 0. \end{aligned}$$

*Due in class on 20th Sept.

Now let's look at the vector, $\mathbf{V} = \Phi \boldsymbol{\nabla} \Phi$ and apply Gauss's divergence theorem to this vector,

$$\int d^3\mathbf{x}\, \boldsymbol{\nabla} \cdot \mathbf{V} = \oiint dS\, \mathbf{n} \cdot \mathbf{V}.$$

The LHS

$$\begin{aligned} \int d^3\mathbf{x}\, \boldsymbol{\nabla} \cdot \mathbf{V} &= \int d^3\mathbf{x}\, \boldsymbol{\nabla} \cdot (\Phi \boldsymbol{\nabla} \Phi) \\ &= \int d^3\mathbf{x} \left(\boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\Phi + \Phi \underbrace{\boldsymbol{\nabla}^2 \Phi}_{=0} \right) \\ &= \int d^3\mathbf{x}\, (\boldsymbol{\nabla}\Phi)^2 , \end{aligned}$$

where we have used Laplace equation to get rid of the second term. Note that this term being a square quantity is manifestly positive semi-definite.
The RHS is,

$$\oiint dS\, \mathbf{n} \cdot \mathbf{V}|_{boundary} = \oiint dS\, \Phi\, (\mathbf{n} \cdot \boldsymbol{\nabla}\Phi) = 0,$$

because either $\Phi = 0$ or $\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}\Phi = 0$ on the boundary since $\Phi_1|_{boundary} = \Phi_2|_{boundary}$ or $\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}\Phi_1|_{boundary} = \hat{\mathbf{n}} \cdot \boldsymbol{\nabla}\Phi_2|_{boundary}$. So Gauss theorem gives us,

$$\int d^3\mathbf{x}\, (\boldsymbol{\nabla}\Phi)^2 = 0.$$

Since the integrand is a square quantity and which is manifestly positive semi-definite while the RHS is zero. Since the integral is actually a "Riemann sum" then with the integrand being a square quantity this means the LHS is a (Riemann) sum of squares. Therefore, the only solution is if each square term in the sum is zero,

$$\boldsymbol{\nabla}\Phi(x) = 0,$$

or,

$$\boldsymbol{\nabla}\Phi_1(x) = \boldsymbol{\nabla}\Phi_2(x),$$

or, integrating both sides,

$$\Phi_1(x) = \Phi_2(x) + c,$$

where c is a constant. However if this is valid for all points including, x on the boundary, then the values are different by an amount, c on the boundary, which is not true by assumption. So $c = 0$ and we have

$$\Phi_1 = \Phi_2,$$

i.e. there can't be two distinct solutions. $\square$

3. **Prove the Mean Value Theorem**. This theorem states that if $\Phi(\mathbf{x})$ satisfies the Laplace equation i.e. for regions outside mass sources, then the value of the potential at any point P

(which can be taken to be the origin of coordinate system without any loss of generality), is given by the mean value of the potential over a sphere (of any radius) which is centered at P.

$$\Phi(\mathbf{x}_P) = \frac{1}{4\pi R^2} \oiint dS \; \Phi(\mathbf{x}'), \tag{1}$$

where dS is the area element (magnitude) on the surface of a sphere of radius R. Note that the sphere is centered at P See equation (1).

Solution:
Let's imagine a region of space bounded by a 2-sphere of radius a. And let's also place the origin of our coordinate system at the center of the sphere which is our point P. Then average value of the gravitational potential over this sphere of radius a, and centered at P:

$$\langle \Phi \rangle_a = \frac{\oiint a^2 \; d\Omega \; \Phi(\mathbf{x})}{4\pi a^2},$$

where $\mathbf{x}$ is the position vector of a point on the surface of the sphere and $d\Omega$ is an infinitesimal solid angle from the origin to an infinitesimal area on the 2-sphere around the point, $\mathbf{x}$. Since a is a constant it can be pulled out of the integral and canceled get canceled out by the factor of a^2 in the denominator, thus giving,

$$\langle \Phi \rangle_a = \frac{1}{4\pi} \oiint d\Omega \; \Phi(\mathbf{x}).$$

This is an equation which at first might look independent of the radius a, but in fact the information about the radius size, a is contained in the position vector, $\mathbf{x}$, to wit, $|\mathbf{x}| = a$. Now we compute the derivative of this averaged potential wrt a,

$$\begin{aligned}
\frac{d\langle \Phi \rangle_a}{da} &= \frac{1}{4\pi} \oiint d\Omega \; \frac{d\Phi(\mathbf{x})}{da} \\
&= \frac{1}{4\pi} \oiint d\Omega \; \left. \frac{\partial \Phi(\mathbf{x})}{\partial r} \right|_{r=a} \\
&= \frac{1}{4\pi} \oiint d\Omega \; \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \Phi,
\end{aligned}$$

where $\hat{\mathbf{n}}$ is the unit normal vector on the surface of the sphere which is radially directed. Further convert this integral over solid angle into a surface integral and then we use Gauss' theorem to convert it into a volume term,

$$\begin{aligned}
\frac{d\langle \Phi \rangle_a}{da} &= \frac{1}{4\pi} \oiint d\Omega \; \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \Phi \\
&= \frac{1}{4\pi a^2} \oiint dS \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \Phi, \quad dS = a^2 d\Omega \\
&= \frac{1}{4\pi a^2} \iiint d^3\mathbf{x} \; \underbrace{\boldsymbol{\nabla}^2 \Phi}_{=0}
\end{aligned}$$

where we have used the "source free" condition, i.e. $\rho = 0$, whereby the potential satisfies Laplace equation, $\boldsymbol{\nabla}^2 \Phi = 0$. Thus we have the result that the mean value over the sphere is independent of the radius,

$$\frac{d\langle \Phi \rangle_a}{da} = 0.$$

i.e. the mean value is identical for distinct values of the radius, say a and ϵ,

$$\langle\Phi\rangle_\epsilon = \langle\Phi\rangle_a.$$

In particular when $\epsilon \to 0$, this mean value turns into the value at the point, p

$$\langle\Phi\rangle_\epsilon = \Phi(P).$$

Thus we have arrived at the mean value theorem,

$$\Phi(P) = \langle\Phi\rangle_a.$$

□

4. Derive the third term in the multipole expansion of the Newton's scalar potential,

$$\Phi(\vec{x}) = -G_N \frac{M}{|\vec{x}|} - G_N \frac{x^i}{|\vec{x}|^3} \int d^3\vec{x'} \rho(\vec{x'})\, x'^i - \frac{1}{2!} G_N \frac{x^i x^j}{|\vec{x}|^5} Q^{ij} + \dots$$

where,

$$Q^{ij} = \int d^3\vec{x'} \rho(\vec{x'})(3x'^i x'^j - \delta^{ij} \vec{x'}^2),$$

is the *quadrupole moment tensor* of the mass distribution. Hint: You will need to write down the Taylor expansion for $\frac{1}{|\vec{x}-\vec{x'}|}$, about $\vec{x'} = 0$, up to first three terms. The first two terms will give the monopole and dipole moments/terms as shown in the lecture, the third piece is the quadrupole moment contribution.

5. Show that the Quadrupole moment tensor of a spherical mass distribution of radius R vanishes. (Hint: Use isotropy/ rotational symmetry but not homogeneity).

6. **Expression of the tidal force in a spaceship**: Consider the situation where we have a spaceship orbiting the earth at a radius, x_0 (i.e. it is the distance between the center of the earth to the center of the spaceship). Now we have two frames, first is the frame fixed on earth with its origin of coordinates at the center of earth. We will assume this to be an inertial frame. The second frame is the frame attached to the spaceship and in which the origin of coordinates is located at the center of mass of the spaceship. This spaceship-frame being accelerated wrt earth-frame, is a non-inertial frame. Show at a point, P in the spaceship whose coordinates are given by $\mathbf{x}$ in the spaceship frame (and say $\mathbf{X}$ in the earth frame), experiences a force due to variation of the earth's gravitational field in the spaceship, given by:

$$f^i = x^j \frac{\partial F^i}{\partial x^j} = -x^j m \left(\frac{\partial^2 \Phi}{\partial x^j \partial x^i} \right) = -x^j mc^2 \, R^i{}_{0j0}.$$

We have introduced this weird looking notation, $R^i{}_{0,j0}$ because it will coincide with Riemann tensor which represents the curvature of the spacetime as a result of gravity.

Solution: The position of the point P from the center of earth frame is,

$$\mathbf{X} = \mathbf{x}_0 + \mathbf{x}$$

and hence, the Newton's second law for this point is,

$$m\ddot{\mathbf{X}} = -\frac{GMm}{|\mathbf{X}|^3}\mathbf{X},$$

or,

$$m\left(\ddot{\mathbf{x}}_{\mathbf{0}} + \ddot{\mathbf{x}}\right) = -\frac{GMm}{|\mathbf{x}_0 + \mathbf{x}|^3}\left(\mathbf{x}_0 + \mathbf{x}\right)$$

To simplify, the rhs we Taylor expand the rhs around $\mathbf{x} = 0$, as distances within the spaceship is much smaller compared to the distance of the (CM of the) space ship from the center of the earth,

$$|\mathbf{x}| \ll |\mathbf{x}_0|$$

and we get to first order,

$$\begin{aligned}
\ddot{\mathbf{x}}_{\mathbf{0}} + \ddot{\mathbf{x}} &= -\frac{GM}{|\mathbf{x}_0|^3}\left(1 - 3\frac{\mathbf{x}_0 \cdot \mathbf{x}}{|\mathbf{x}_0|^2} + O\left(\frac{|\mathbf{x}|^2}{|\mathbf{x}_0|^2}\right)\right)(\mathbf{x}_0 + \mathbf{x}) \\
&= -\frac{GM}{|\mathbf{x}_0|^3}\mathbf{x}_0 - \frac{GM}{|\mathbf{x}_0|^3}\mathbf{x} + 3\frac{GM}{|\mathbf{x}_0|^3}\frac{\mathbf{x}_0 \cdot \mathbf{x}}{|\mathbf{x}_0|^2}\mathbf{x}_0 + O\left(\frac{|\mathbf{x}|^2}{|\mathbf{x}_0|^2}\right).
\end{aligned}$$

But the spaceship is itself orbiting under the gravitational centripetal force,

$$\ddot{\mathbf{x}}_{\mathbf{0}} = -\frac{GM}{|\mathbf{x}_0|^3}$$

So within in the spaceship the relative acceleration of the point P (the tidal acceleration) is,

$$\begin{aligned}
\ddot{\mathbf{x}} &= (\ddot{\mathbf{x}}_{\mathbf{0}} + \ddot{\mathbf{x}}) - \ddot{\mathbf{x}}_{\mathbf{0}} \\
&= -\frac{GM}{|\mathbf{x}_0|^3}\mathbf{x} + 3\frac{GM}{|\mathbf{x}_0|^3}\frac{\mathbf{x}_0 \cdot \mathbf{x}}{|\mathbf{x}_0|^2}\mathbf{x}_0
\end{aligned}$$

or in terms of components, the tidal acceleration

$$\begin{aligned}
\ddot{x}^i &= -\frac{GM}{|\mathbf{x}_0|^3}x^i + 3\frac{GM}{|\mathbf{x}_0|^3}\frac{x_0^j x^j}{|\mathbf{x}_0|^2}x_0^i \\
&= -\left(\frac{GM}{|\mathbf{x}_0|^3}\delta^{ij} - 3\frac{GM}{|\mathbf{x}_0|^5}x_0^i x_0^j\right)x^j. \qquad (2)
\end{aligned}$$

Now the gravitational scalar potential is,

$$\Phi(P) = -\frac{GM}{|\mathbf{x}_0 + \mathbf{x}|}$$

and hence,

$$\begin{aligned}
\frac{\partial \Phi}{\partial x^i} &= \frac{GM}{|\mathbf{x}_0 + \mathbf{x}|^3}(x_0^i + x^i), \\
\Rightarrow \frac{\partial^2 \Phi}{\partial x^i \partial x^j} &= -3\frac{GM}{|\mathbf{x}_0 + \mathbf{x}|^5}(x_0^j + x^i)(x_0^i + x^i) + \delta^{ij}\frac{GM}{|\mathbf{x}_0 + \mathbf{x}|^3} \\
\Rightarrow \left.\frac{\partial^2 \Phi}{\partial x^i \partial x^j}\right|_{x=0} &= -3\frac{GM}{|\mathbf{x}_0|^5}x_0^i x_0^j + \frac{GM}{|\mathbf{x}_0|^3}\delta^{ij}. \qquad (3)
\end{aligned}$$

From the expression of the tidal acceleration, (2) and expression of the second derivative of the scalar potential around the center of the spaceship (3) , we get the tidal acceleration expression in terms of potential,

$$\ddot{x}^i = -\left.\frac{\partial^2 \Phi}{\partial x^i \partial x^j}\right|_{x=0} .$$

Multiplying both sides by the mass of the particle at P, i.e. m we have the tidal force,

$$f^i = mx^i = -m \left.\frac{\partial^2 \Phi}{\partial x^i \partial x^j}\right|_{x=0} .$$