PH 6458/EP 4258: Gravitation and Cosmology (Fall 2019) Homework Set 5*

October 27, 2019

1. **Homework:** Consider two dimensional flat space also known as 2d Euclidean space, $\mathbb{R}^2$. This is a trivial manifold because it can be covered entirely by a single chart. One can either use the *Cartesian* coordinates to label points in $\mathbb{R}^2$ i.e. (x, y), or one can use the plane polar coordinate system (r, θ) to label (almost) all the points in $\mathbb{R}^2$. The distance between two points in $\mathbb{R}^2$ say (r, θ) and $(r+dr, \theta+d\theta)$ is given by the familiar "squared line element",

$$ds^2 = dr^2 + r^2\, d\theta^2.$$

So the metric, g is given by the components,

$$g_{rr} = 1,\ g_{\theta\theta} = r^2,\ g_{r\theta} = g_{\theta r} = 0.$$

Check that under the coordinate transformations,

$$\begin{aligned} x &= r\,\cos\theta, \\ y &= r\,\sin\theta, \end{aligned}$$

the metric transforms to,

$$ds^2 = dx^2 + dy^2.$$

(Hint: Use the transformation law for metric tensor).

(4)

2. Show that adding a $(1,2)$-type tensor, $T^\mu_{\nu\lambda}$ to an affine connection, $\Gamma^\mu_{\nu\lambda}$ gives us another new affine connection, $\bar{\Gamma}^\mu_{\nu\lambda}$ i.e. show that,

$$\bar{\Gamma}^\mu_{\nu\lambda} \equiv \Gamma^\mu_{\nu\lambda} + T^\mu_{\nu\lambda}$$

also transforms like an affine connection under change of coordinate charts. An equivalent statement is: The difference of two affine connections is a (1,2) rank pure tensor.

(3)

3. **Show that** the torsion-free and metric compatibility conditions full determine the connection in terms of the metric and its first derivatives, namely, the Christoffel symbols,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right).$$

(4)

*Due in class on Thursday, Oct. 31

4. Compute the Riemann tensor components for a two-dimensional manifold, namely the 2-sphere of radius R, given by the metric in polar coordinate chart,

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

You will first need to compute, the Levi-Civita connection, $\Gamma^{\mu}_{\nu\lambda}$ and then plug them in the expression for the Riemann Curvature in terms of connection and it's first order derivatives. (Hint: There is just *one* independent component of the Riemann tensor in 2d space!)

($4 + 3 = 7$)

5. Derive the following identities for a emphtorsion-free connection:
i. $[\nabla_\mu, \nabla_\nu]\,\varphi = 0$
ii. $[\nabla_\mu, \nabla_\nu]\,V^\lambda = R^\lambda{}_{\rho\mu\nu}\,V^\lambda$
iii. $[\nabla_\mu, \nabla_\nu]\,\omega_\lambda = -R^\rho{}_{\lambda\mu\nu}\omega^\lambda$,
Based on the above pattern can you guess the RHS of $[\nabla_\mu, \nabla_\nu]\,T^{\mu_1\cdots\mu_p}_{\nu_1,\ldots,\nu_q}$?

($2 + 4 + 4 + 2 = 12$)