PH 6458/EP 4258: Gravitation and Cosmology (Fall 2019)
Notes from Monday Oct. 10 lecture

October 29, 2019

1 Einstein-Hilbert action

The Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N \, T_{\mu\nu},$$

can be derived from an action principle. The action was first proposed by David Hilbert and it reads as,

$$I_{EH} = \frac{1}{16\pi G_N} \int d^4x \, \sqrt{-g} \, R + I_M$$

where I_M is the matter action. Now we vary the action by varying the metric, $\delta g^{\mu\nu}$,

$$\delta I_{EH} = \frac{1}{16\pi G_N} \int d^4x \, \left(\delta\sqrt{-g} \, R + \sqrt{-g} \, \delta R \right) + \int d^4x \, \delta g^{\mu\nu} \, \frac{\delta I_M}{\delta g^{\mu\nu}}.$$

Let's look at each of the terms,

$$\begin{aligned} \delta\sqrt{-g} &= \frac{-\delta g}{2\sqrt{-g}} \\ &= \frac{-g g^{\mu\nu} \delta g_{\mu\nu}}{2\sqrt{-g}} \\ &= \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} \\ &= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \end{aligned}$$

Here we have used the identity for n-dimensional metric matrix,

$$\begin{aligned} \delta g &= \delta \left(\frac{1}{n!} \varepsilon^{\mu_1\mu_2\ldots\mu_n} \, \varepsilon^{\nu_1\nu_2\ldots\nu_n} \, g_{\mu_1\nu_1} \, g_{\mu_2\nu_2} \, \ldots g_{\mu_n\nu_n} \right) \\ &= \left(\frac{1}{n!} \varepsilon^{\mu_1\mu_2\ldots\mu_n} \, \varepsilon^{\nu_1\nu_2\ldots\nu_n} \, g_{\mu_2\nu_2} \, \ldots g_{\mu_n\nu_n} \right) \delta g_{\mu_1\nu_1} + \left(\frac{1}{n!} \varepsilon^{\mu_1\mu_2\ldots\mu_n} \, \varepsilon^{\nu_1\nu_2\ldots\nu_n} \, g_{\mu_1\nu_1} \, g_{\mu_3\nu_3} \ldots g_{\mu_n\nu_n} \right) \delta g_{\mu_2\nu_2} + \ldots \\ &= \frac{1}{n} \left(\underbrace{\frac{1}{(n-1)!} \varepsilon^{\mu_1\mu_2\ldots\mu_n} \, \varepsilon^{\nu_1\nu_2\ldots\nu_n} \, g_{\mu_2\nu_2} \, \ldots g_{\mu_n\nu_n}}_{=\mathrm{Cof}(g_{\mu_1\nu_1})} \right) \delta g_{\mu_1\nu_1} + \ldots \\ &= \mathrm{Cof}\left(g_{\mu\nu}\right) \, \delta g_{\mu\nu} \\ &= g \, g^{\mu\nu} \, \delta g_{\mu\nu}. \end{aligned}$$

Next consider the variation of the second term,

$$\delta R = \delta\left(R_{\mu\nu} g^{\mu\nu}\right) = \delta R_{\mu\nu} g^{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu}.$$

Now,

$$\begin{aligned}
\delta R_{\mu\nu} = \delta R^{\lambda}{}_{\mu\lambda\nu} &= \delta\partial_\lambda \Gamma^\lambda_{\nu\mu} - \delta\partial_\nu \Gamma^\lambda_{\lambda\mu} + \delta\left(\Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\rho\lambda}\right) - \delta\left(\Gamma^\lambda_{\rho\mu}\Gamma^\rho_{\nu\lambda}\right) \\
&= \partial_\lambda\left(\delta\Gamma^\lambda_{\nu\mu}\right) - \partial_\nu\left(\delta\Gamma^\lambda_{\lambda\mu}\right) + \delta\left(\Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\rho\lambda}\right) - \delta\left(\Gamma^\lambda_{\rho\mu}\Gamma^\rho_{\nu\lambda}\right) \\
&= \nabla_\lambda\left(\delta\Gamma^\lambda_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^\lambda_{\lambda\mu}\right).
\end{aligned}$$

Here we have used the fact that $\delta\Gamma^\lambda_{\nu\mu}$ being the (infinitesimal) difference between two affine connections is a rank $(1,2)$ tensor. Thus we have,

$$\delta R = \nabla_\lambda\left(\delta\Gamma^\lambda_{\nu\mu} g^{\mu\nu}\right) - \nabla^\mu\left(\delta\Gamma^\lambda_{\lambda\mu}\right) + R_{\mu\nu}\delta g^{\mu\nu}.$$

Finally the full variation of the action,

$$\begin{aligned}
\delta I_{EH} &= \frac{1}{16\pi G_N}\int d^4x\ \left(\delta\sqrt{-g}\, R + \sqrt{-g}\,\delta R\right) + \int d^4x\ \delta g^{\mu\nu}\ \frac{\delta I_M}{\delta g^{\mu\nu}}. \\
&= \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\ \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R + 8\pi G_N \frac{2}{\sqrt{-g}}\frac{\delta I_M}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu} \\
&\qquad + \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,\underbrace{\left[\nabla_\lambda\left(\delta\Gamma^\lambda_{\nu\mu}g^{\mu\nu}\right) - \nabla^\mu\left(\delta\Gamma^\lambda_{\lambda\mu}\right)\right]}_{\textit{total covariant derivative}} \\
&= \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\ \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R + 8\pi G_N \frac{2}{\sqrt{-g}}\frac{\delta I_M}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu} + surface \quad terms
\end{aligned} \tag{1}$$

Thus demanding $\delta I_{EH} = 0$ for arbitrary $\delta g^{\mu\nu}$ as well as by demanding the surface terms vanish[1], implies that the integrand must vanish i.e.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R + 8\pi G_N\left(\frac{2}{\sqrt{-g}}\frac{\delta I_M}{\delta g^{\mu\nu}}\right) = 0,$$

we have the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R = 8\pi G_N T_{\mu\nu}.$$

Here we have identified the matter stress tensor,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta I_M}{\delta g^{\mu\nu}}. \tag{2}$$

One can easily check that the this indeed reproduces the symmetric stress tensor, for example using the scalar field action (in curved background)

$$I_M = \int d^4x\sqrt{-g}\ \left[-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right] = \int d^4x\sqrt{-g}\ \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right]$$

[1]For manifolds with boundaries these surface terms might not vanish, and we will have to include additional boundary terms in the Hilbert action e.g. the Gibbons-Hawking-York (GHY) term, so that these surface terms cancel with the variation of the GHY boundary term(s).

Varying this as a result of varying the metric

$$\begin{aligned}
\delta I_M &= \int d^4x \, \left(\delta\sqrt{-g}\mathcal{L}_\varphi - \sqrt{-g}\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \right) \\
&= \int d^4x \, \left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\mathcal{L}_\varphi - \sqrt{-g}\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \right) \\
&= \int d^4x \, \delta g^{\mu\nu} \, \left(-\frac{1}{2}\sqrt{-g} \right) \left(g_{\mu\nu}\mathcal{L}_\varphi + \partial_\mu\varphi\partial_\nu\varphi \right), \\
\Rightarrow \frac{-2}{\sqrt{-g}}\frac{\delta I_M}{\delta g^{\mu\nu}} &= g_{\mu\nu}\mathcal{L}_\varphi + \partial_\mu\varphi\partial_\nu\varphi = T_{\mu\nu}.
\end{aligned}$$

Thus we reproduce the stress tensor of the scalar field in Minkowski spacetime once we set $g_{\mu\nu} = \eta_{\mu\nu}$.

Remarks

- For manifolds with boundaries these surface terms might not vanish, and we will have to include additional boundary terms in the Hilbert action e.g. the Gibbons-Hawking-York (GHY) counterterm such that the variation of the GHY boundary term(s) cancel the surface terms in (1). The GHY term for non-null boundaries look like,

  $$I_{GHY} = \frac{1}{8\pi G_N}\int_{\partial M}\sqrt{|h|}K,$$

  where h_{ab} is the metric induced on the boundary ∂M, and K is the trace of the second fundamental form or extrinsic curvature,

  $$K = \nabla_\mu n^\mu,$$

  where n^μ is the unit outward normal at the boundary.

- The Einstein-Hilbert action can be simply be written down by demanding the action be diffeomorphism invariant (general coordinate invariant), real, contain at best two derivatives of the metric field. Since the action is a spacetime integral, one must have the invariant volume element,

  $$I \sim \int d^4x\sqrt{-g}\,\left(\ldots\right)$$

  One cannot have quadratic terms of first order derivatives, $\nabla_\mu g_{\alpha\beta}\,\nabla^\mu g^{\alpha\beta}$, because metric compatibility means these terms are zero. Thus we are forced to construct an invariant/scalar made up of double derivatives of the metric. The only such choice is the Ricci scalar, R. Thus we arrive at the form,

  $$I \sim \int d^4x\sqrt{-g}\,R.$$

  The proportionality constant can then be fixed up to a dimensionless number by making the action is dimensionless.

  $$I \sim \frac{1}{G_N}\int d^4x\sqrt{-g}\,R.$$

Exercise: Write down the action for a Maxwell field (in the absence of charges) in curved spacetime. Find out the stress tensor for the EM field using the formula (2). Check that it indeed matches/reduces to the expression for the symmetric stress tensor for the Maxwell field in Minkowski space once you set $g_{\mu\nu} = \eta_{\mu\nu}$.