PH 6458/ EP 4258: The Schwarzschild Solution and Precession of Perihelion

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The Schwarzschild solution describes the gravitational field (metric) produced by a spherical mass distribution such as the gravitational field of the stars such as the Sun or planets such as the earth. The Schwarzschild metric is an exact solution to the full non-linear Einstein field equations, in fact the first non-trivial solution in general relativity to be written down. This is quite unlike for some other well known solutions, say, gravitational waves, which are solutions to linearized Einstein field equations (namely the Fierz-Pauli equation) and hence only represent an *approximate* solution (when the gravitational field is weak). In today's lecture we review the Schwarzschild solution and then we examine the effects of the perturbations to Newtonian theory of planetary dynamics due to general relativity.

1 Ricci-flat metrics

Since we are interested in writing down the metric at a point in the gravitational field which is outside the spherical mass distribution, we will set $T_{\mu\nu} = 0$ at that field point i.e. the RHS of the field equations vanish,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{1}$$

Taking a trace of this equation (i.e. contracting both sides by $g_{\mu\nu}$), we get,

$$R = 0,$$

and inserting this back in the field equation (1), we get,

$$R_{\mu\nu} = 0.$$

Thus at a point in a gravitational field outside the matter source, one must have the Ricci tensor vanishing. Such metrics obtained as solutions describing gravitational field in regions where there is no matter are known as *vacuum solutions* to general relativity/ Einstein field equations and as we have just seen these metrics are **Ricci-flat** metrics (i.e. a metric for which the Ricci tensor vanishes). Evidently, one such solution is when the components of the *Riemann* tensor, $R_{\mu\nu\rho\sigma}$, itself vanishes, because the Ricci tensor is given by a particular trace of the Riemann tensor, $R_{\mu\nu} \equiv g^{\rho\sigma} R_{\rho\mu\sigma\nu}$. What is this solution for which the Riemann curvature vanishes? It has to be the Minkowski metric! Recall this metric describes a solution where there is no gravitational field at all, i.e. no matter sources anywhere. However, general Ricci-flat metrics are more interesting than empty space (Riemann flat metric) because Ricci flat solutions describe spacetimes where there is non-zero gravitational field.

2 The Metric Ansatz

Our job here is to solve for a 4-dimensional Ricci-flat metric which has a spherical symmetry. Now usually in physics when we are obtaining solutions outside sources, e.g. electric field outside a spherical charge distribution, we can choose coordinates appropriate to the problem and then write down the Maxwell or Poisson/Laplace equation in that coordinate system. However in general relativity one cannot choose a metric and a coordinate system a priori because the coordinate system and the metric itself should be obtained as a part of the solution. This situation in gravity is thus different from other field equations in physics. So we have to start with the four coordinates labeling our metric (spacetime) by abstract labels x^0, x^1, x^2, x^3 and the metric components, are then some yet to be determined functions of these four coordinates, $g_{\mu\nu}(x^0, x^1, x^2, x^3)$. Since we are dealing with abstract labels, x^0, x^1, x^2, x^3, one can ask what do we mean when we say our metric should have spherical symmetry in terms of such abstract coordinates? The answer is that the metric is such that, at least a subgroup of isometry group should be that of a sphere (in 4 dimensions, we mean the 2-sphere) i.e. $SO(3)$. Then an important theorem in Riemannian geometry (Frobenius theorem) tells us that we can foliate our spacetime into 2-dimensional foils/sheets described two coordinates, say x^2, x^3, which are the same as those parametrizing a sphere namely, $x^2 = \theta,\ x^3 = \phi$ and the metric on these foliations/sheets are given by that of the 2-sphere up to a conformal prefactor[1] ,

$$ds^2|_{x^0,x^1} = C(x^0, x^1)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{2}$$

where the prefactor, $C(x^0, x^1)$, is a function of the two other coordinates (i.e not θ, ϕ). The notation, $ds^2|_{x^0,x^1}$ evidently means we are looking at foils/leaves when these other two coordinates x^0, x^1 are held constant. We will from now on denote the 2-sphere metric by the notation,

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

Also from now on we will rename, $x^0 = t$ and $x^1 = r$, where we have kept in mind that out metrics are Lorentzian, i.e. has one time and 3 space coordinates. An ansatz for the full 4-dimensional metric can then be written down,

$$\begin{aligned} ds^2 = &-A(t,r)\,dt^2 + B(t,r)\,dr^2 + 2D(t,r)\,drdt + C(t,r)\,d\Omega_2^2 \\ &+ 2E(t,r)\,dtd\theta + 2F(t,r)\,dtd\varphi + 2G(t,r)\,drd\theta + 2H(t,r)\,drd\varphi. \end{aligned}$$

It can be easily checked that when t, r are held constant, i.e. $dt = dr = 0$, this gives us back (2). At this point one can ask why did we not take the metric coefficients,$A, ..., H$ to be function of θ, ϕ as well. The reason is that if these coefficients depend on angles i.e. change with direction then the spacetime will not be isotropic/rotation invariant. Similarly we can infer that, $E, F, G, H = 0$ because these terms tell us that metric (squared distance) is different if we change θ or ϕ in the clockwise or anticlockwise direction i.e. these terms depend on the sign of $d\theta$ and/or $d\phi$. Thus, our ansatz simplifies a lot,

$$ds^2 = -A(t,r)\,dt^2 + B(t,r)\,dr^2 + 2D(t,r)\,drdt + C(t,r)\,d\Omega_2^2.$$

Next we make a coordinate transformation i.e. define a new coordinate $\tilde{r}^2 = C(t,r)$ and eliminate r in favor of $\tilde{r}$, i.e. by replacing, $dr = \frac{\partial r}{\partial \tilde{r}}d\tilde{r} + \frac{\partial r}{\partial t}dt$ etc.,

$$ds^2 = -\tilde{A}(t,\tilde{r})\,dt^2 + \tilde{B}(t,\tilde{r})\,d\tilde{r}^2 + 2\tilde{D}(t,\tilde{r})\,d\tilde{r}dt + \tilde{r}^2\,d\Omega_2^2.$$

[1] See Carroll's notes (Chapter 7) for the details

The functions $\tilde{A}, \tilde{B}, \ldots$ are now new (unknown) functions of $t, \tilde{r}$.

Completing the squares, this metric can be turned into,

$$ds^2 = -\tilde{A}(t,\tilde{r})\left(dt - \frac{\tilde{D}(t,\tilde{r})}{\tilde{A}(t,\tilde{r})}d\tilde{r}\right)^2 + \left(\tilde{B}(t,\tilde{r}) + \frac{\tilde{D}^2(t,\tilde{r})}{\tilde{A}(t,\tilde{r})}\right)d\tilde{r}^2 + \tilde{r}^2\, d\Omega_2^2.$$

One can find an integrating factor for the differential, $dt - \frac{\tilde{D}(t,\tilde{r})}{\tilde{A}(t,\tilde{r})}d\tilde{r}$, such that

$$d\tilde{t} = F(t,\tilde{r})\left(dt - \frac{\tilde{D}(t,\tilde{r})}{\tilde{A}(t,\tilde{r})}d\tilde{r}\right)$$

is an exact differential. Integrating this one can substitute $\tilde{t} = \tilde{t}(t,r)$ and thus eliminate,t in the metric ansatz in favor of this new coordinate,$\tilde{t}$,

$$ds^2 = -\frac{\tilde{A}(t,\tilde{r})}{F(t,\tilde{r})}d\tilde{t}^2 + \left(\tilde{B}(t,\tilde{r}) + \frac{\tilde{D}^2(t,\tilde{r})}{\tilde{A}(t,\tilde{r})}\right)d\tilde{r}^2 + \tilde{r}^2\, d\Omega_2^2.$$

Since the exact forms of the metric coefficients, $\tilde{A}, ..., F$ are undetermined we are free to call these functions anything. Thus to reduce cumber we relabel them,

$$\frac{\tilde{A}(t,\tilde{r})}{F(t,\tilde{r})} \to A(\tilde{t},\tilde{r}), \tilde{B}(t,\tilde{r}) + \frac{\tilde{D}^2(t,\tilde{r})}{\tilde{A}(t,\tilde{r})} \to B(\tilde{t},\tilde{r}),$$

and metric ansatz looks cleaner,

$$ds^2 = -A(\tilde{t},\tilde{r})d\tilde{t}^2 + B(\tilde{t},\tilde{r})d\tilde{r}^2 + \tilde{r}^2\, d\Omega_2^2.$$

Next we relabel $\tilde{t}, \tilde{r}$ to t, r as these are nothing but abstract labels,

$$ds^2 = -A(t,r)\, dt^2 + B(t,r)\, dr^2 + r^2 d\Omega_2^2.$$

Finally since A, B must be positive functions, we express them in a way which will be convenient when it comes to actually solving the Ricci equation, to wit, $A(t,r) = e^{a(t,r)}$ and $B(t,r) = e^{b(t,r.)}$,

$$ds^2 = -e^{a(t,r)}\, dt^2 + e^{b(t,r.)}\, dr^2 + r^2 d\Omega_2^2. \tag{3}$$

Since the Einstein equation are differential equations, we will need some boundary conditions as well as initial conditions. To arrive at these boundary conditions we use the physical requirement that far from the spherical source the gravitational field should vanish and the metric should turn Minkowski,

$$ds^2|_{r\to\infty} = -dt^2 + dr^2 + r^2 d\Omega_2^2,$$

which imply,

$$\lim_{r\to\infty} a(t,r), b(t,r) = 0. \tag{4}$$

As we will see we do not need to specify any initial conditions i.e. boundary conditions in time.

3 Solving the Ricci Equation

To this end we first compute the Christoffel symbols and Ricci tensor components using their definitions,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(g_{\rho\nu,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho} \right),$$

$$R_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}_{\nu\mu} - \partial_\nu \Gamma^{\lambda}_{\lambda\mu} + \Gamma^{\rho}_{\nu\mu} \Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\lambda}_{\rho\mu} \Gamma^{\rho}_{\lambda\nu}.$$

the metric ansatz, (2). The nonvanishing Christoffel symbols are listed below,

$$\Gamma^{t}_{tt} = \frac{\dot{a}}{2}, \qquad \Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \frac{a'}{2}, \qquad \Gamma^{t}_{rr} = \frac{\dot{b}}{2} e^{b-a},$$

$$\Gamma^{r}_{tt} = \frac{a'}{2} e^{a-b}, \qquad \Gamma^{r}_{tr} = \Gamma^{r}_{rt} = \frac{\dot{b}}{2}, \qquad \Gamma^{r}_{rr} = \frac{b'}{2}, \qquad \Gamma^{r}_{\theta\theta} = -r e^{-b}, \qquad \Gamma^{r}_{\phi\phi} = -r \sin^2\theta e^{-b},$$

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}, \qquad \Gamma^{\theta}_{\phi\phi} = -\sin\theta \, \cos\theta,$$

$$\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}, \qquad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta.$$

where $\dot{} = \partial_t$ and $' = \partial_r$. These Christoffel symbols will be useful when we look at the geodesic equation. From these the nonvanishing components of the Ricci tensor are,

$$R_{tt} = e^{a-b} \left(\frac{a''}{2} - \frac{a'b'}{4} + \frac{a'}{r} + \frac{a'^2}{4} \right) - \frac{\ddot{b}}{2} + \frac{\left(\dot{a} - \dot{b}\right)\dot{b}}{4},$$

$$R_{tr} = \frac{\dot{b}}{r},$$

$$R_{rr} = -\frac{a''}{2} - \frac{a'^2}{4} + \frac{a'\,b'}{4} + \frac{b'}{r} + e^{b-a} \left[\frac{\dot{b}}{2} - \frac{\left(\dot{a} - \dot{b}\right)\dot{b}}{4} \right],$$

$$R_{\theta\theta} = 1 + e^{-b} \left(\frac{b' - a'}{2} r - 1 \right),$$

$$R_{\phi\phi} = \sin^2\theta \left[1 + e^{-b} \left(\frac{b' - a'}{2} r - 1 \right) \right].$$

From the above expressions, it is evident that the easiest to solve is, $R_{tr} = 0$:

$$\frac{\dot{b}}{r} = 0 \Rightarrow b = b(r)$$

i.e. b is purely a function of r since its t-derivative vanishes. Next we observe that R_{tt} and R_{rr} have a lot of similar terms and there is a potential to cancel out and give us something simple. We identify the combination,

$$R_{rr} + e^{b-a} R_{tt} = 0,$$

which leads to the equation,

$$\frac{a' + b'}{r} = 0 \Rightarrow a(t, r) = -b(r) + f(t)$$

where $f(t)$ is a constant of r-integration. Next we look at the $R_{\theta\theta} = 0$ equation, in the expression for $R_{\theta\theta}$ we substitute the above expression for $a(t, r)$, to get,

$$\begin{aligned} 1 + e^{-b}\left(r\, b' - 1\right) &= 0, \\ \Rightarrow 1 - \frac{d}{dr}\left(r\, e^{-b(r)}\right) &= 0 \\ \Rightarrow e^{b(r)} &= \frac{1}{1 + \frac{C}{r}}, \end{aligned}$$

where C is the constant of integration. This automatically satisfies the boundary condition, $b(r) \to 0$ as $r \to \infty$. Demanding $\lim_{r\to\infty} a(t, r) = 0$ then implies, $f(t) = 0$. Thus we have solved the Schwarzschild metric up to a constant of integration, C,

$$ds^2 = -\left(1 + \frac{C}{r}\right) dt^2 + \frac{dr^2}{1 + \frac{C}{r}} + r^2\, d\Omega_2^2. \tag{4}$$

We observe a remarkable fact, the time-dependence has gone away! Spherical symmetry gets rid of the off-diagonal terms like $dt\, dx^{1,2,3}$ *and* forces the metric coefficients to be time-independent[2]. Such a metric which is time-independent and orthogonal to constant t slices, i.e. $g_{tx} = 0$ for any spatial coordinate x is called a ***static*** metric.

To fix the constant of integration, C, we look at the Newtonian limit of the Schwarzschild metric. We know that in the Newtonian limit (weak fields/linear and small particle velocities),NW

$$g_{00} = \eta_{00} - 2\Phi$$

where Φ is the Newton's scalar potential. For a spherical massive object, $\Phi = -\frac{G_N M}{r}$, so we have,

$$C = -2G_N\, M.$$

(In units where speed of light is *not* set to unity, $C = -\frac{2G_N M}{c^2}$). Thus we arrive at the final version of the Schwarzschild metric,

$$\boxed{ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2G_N M}{r}} + r^2\, d\Omega_2^2} \tag{5}$$

It is interesting consider the example of a spherical shell. In Newtonian gravity we know that the gravitational field outside the shell is same as if the entire mass was concentrated at the center of the sphere while inside the shell, the gravitational field vanishes - a result known as Newton's theorem. Similarly one might ask what is the situation in general relativity, i.e. what is the metric outside and inside a spherical shell. The metric outside the shell as we have established is given by the Schwarzschild metric (5). Inside we expect a similar solution (4) to hold but possibly with a different constant of integration, C. However inside the shell, specifically at the origin/center of the shell, $r = 0$ there is no matter, but the metric (4) diverges for any non-zero value of C. Thus

[2] This celebrated result goes by the name of Birkhoff's theorem

we conclude that inside the shell we have to set, $C = 0$ and the metric inside the shell is same as the Minkowski (flat) metric.

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2.$$

Since a flat metric implies the Riemann curvature tensor vanishing, there is no gravitational field inside the shell. Thus Newton's theorem holds even in general relativity.

Comment: The parameter, M in general relativity is the *total* energy in the entire spacetime, i.e. the mass-energy of the matter as well as the energy contained in the gravitational field. In the Newtonian limit of course the energy in the gravitational field is so small compared to the matter source that we can virtually identify M with the mass of the source.

4 Precession of Perihelion of planets

Now we look at trajectories of point particles in the Schwarzschild geometry, which applies to planets orbiting a central mass such as the Sun. At leading order the geodesics, as expected, will coincide with Newtonian theory i.e. orbits will be ellipses, while the first post-Newtonian correction will lead to interesting deviations, namely precession of perihelion of the elliptic orbits. For Mercury, this precession was observed to be 43 arcsecs per century, which was successfully reproduced by the first post-Newtonian correction of general relativity. The geodesic equation is,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0,$$

τ being the proper time. However one should bear in mind that the velocities are not all independent and are related as follows from the definition of proper time,

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= -d\tau^2 \\ g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} &= -1. \end{aligned} \tag{6}$$

Since none of the metric coefficients depend on time, t, we can without confusion use a dot to denote proper time derivatives $\frac{dx^\mu}{d\tau} = \dot{x}^\mu$. Using the Christoffel symbols, the geodesic equations are,

$$\begin{gathered} \ddot{t} + a'\,\dot{r}\dot{t} = 0, \\ \ddot{r} + \frac{a'}{2}e^{a-b}\dot{t}^2 + \frac{b'}{2}\dot{r}^2 - re^{-b}\dot{\theta}^2 - r\sin^2\theta e^{-b}\,\dot{\phi}^2 = 0, \\ \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0, \\ \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0. \end{gathered}$$

First we will solve the θ equation. Without loss of generality, we can assume the initial conditions $\theta(0) = \frac{\pi}{2}$ and $\dot{\theta}(0) = 0$, i.e. the plane defined by the position vector of the point particle from the central mass as the origin and the spatial velocity vector at $\tau = 0$ to be the equatorial plane.

Plugging these in the θ-equation gives us, $\ddot{\theta}(0) = 0$. Taking further derivatives of the θ-equation and then plugging $\tau = 0$, one can check that,

$$\theta^{(n)}(0) = 0, \forall n \geq 1.$$

Then one can use the Taylor series formula,

$$\theta(\tau) = \theta(0) + \dot{\theta}(0)\,\tau + \ddot{\theta}(0)\,\frac{\tau^2}{2!} + \dddot{\theta}(0)\,\frac{\tau^3}{3!} + \ldots$$

to obtain,

$$\theta(\tau) = \frac{\pi}{2}. \tag{7}$$

Thus the motion is confined entirely to the equatorial plane at all times. This is not a surprising result, the same result holds in the Newtonian theory as well. We will substitute $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$ (for general τ) in the other equations to simplify and solve them. The ϕ equation becomes,

$$\begin{aligned} \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\theta} &= 0 \\ \Rightarrow r^2\ddot{\phi} + 2r\dot{r}\dot{\phi} &= 0, \\ \Rightarrow \frac{d}{d\tau}\left(r^2\dot{\phi}\right) &= 0 \\ \Rightarrow r^2\dot{\phi} &= h. \end{aligned} \tag{8}$$

The constant h is actually a conserved charge, namely the angular momentum (divided by the particle mass). Similarly, the t-equation leads to a conserved charge,

$$\begin{aligned} \ddot{t} + a'\,\dot{r}\dot{t} &= 0 \\ \Rightarrow \ddot{t} + \frac{d}{d\tau}a(r)\dot{t} &= 0 \\ \Rightarrow \ddot{t} + \frac{1}{e^{a(r)}}\frac{d}{d\tau}\left(e^{a(r)}\right)\dot{t} &= 0 \\ \Rightarrow e^a\ddot{t} + \frac{d}{d\tau}\left(e^{a(r)}\right)\dot{t} &= 0 \\ \Rightarrow \frac{d}{d\tau}\left(e^a\dot{t}\right) &= 0 \\ \Rightarrow e^a\dot{t} &= \varepsilon. \end{aligned}$$

This constant of integration is another conserved charge, namely the energy of the point particle per unit mass. One can easily see this by going far from the central mass, i.e. when $r \to \infty$, when $a = 0$ and $\varepsilon = \frac{dt}{d\tau} = \gamma = \frac{E}{m}$. Now we are left with just solving the radial equation. However that is a second order differential equation and we will instead solve for the radial coordinate by solving a first order differential equation, namely the constraint, (6)

$$\begin{aligned} -\left(1 - \frac{2G_NM}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{2G_NM}{r}} + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 &= -1 \\ \Rightarrow -\frac{\varepsilon^2}{1 - \frac{2G_NM}{r}} + \frac{\dot{r}^2}{1 - \frac{2G_NM}{r}} + \frac{h^2}{r^2} &= -1 \end{aligned}$$

Further, changing variables, $u = \frac{1}{r}$ and changing the affine parameter from τ to ϕ,i.e. from $r(\tau)$ to $u(\phi)$, we get,

$$h^2\left(\frac{du}{d\phi}\right)^2 - \varepsilon^2 + h^2u^2\left(1 - 2G_N M u\right) = -1 + 2G_N M u,$$

$$\left(\frac{du}{d\phi}\right)^2 - \frac{2G_N M}{h^2}u + u^2 - 2G_N M u^3 = \frac{\varepsilon^2 - 1}{h^2}.$$

Taking a derivative, $\frac{d}{d\phi}$ of this equation, gives us a new equation,

$$\frac{d^2u}{d\phi^2} - \frac{G_N M}{h^2} + u - 3G_N M u^2 = 0 \tag{9}$$

Comparing this with the Newtonian theory given by,

$$\frac{d^2u}{d\phi^2} - \frac{G_N M}{h^2} + u = 0 \tag{10}$$

we identify the general relativistic correction term to be $-3G_N M u^2$, where the dimensionless perturbation parameter is $3G_N M u$ or in units where speed of light is not unity, $\frac{3G_N M u}{c^2}$. It is clear that this parameter will be largest for the innermost planet for which $u = \frac{1}{r}$ is maximum. For instance, consider the solar system, for which, $G_N \sim 10^{-11}$, $M \sim 10^{30} Kg$, $c^2 \sim 10^{16} m/s$, while $u \sim 10^{-10} m^{-1}$,we compute

$$\frac{3G_N M u}{c^2} \sim 10^{-7}$$

which is a extremely small compared to unity. Hence we will consider the perturbation to linear order only,

$$u = u^{(0)} + u^{(1)},$$

and then the leading order of (9) is,

$$\frac{d^2u^{(0)}}{d\phi^2} - \frac{G_N M}{h^2} + u^{(0)} = 0,$$

same as that of Newton's theory. The solution to these are ellipses,

$$u^{(0)} = \frac{G_N M}{h^2}\left[1 + e\,\cos\left(\phi - \phi_P\right)\right].$$

Here e is the eccentricity of the ellipse while ϕ_p is evidently the perihelion. The subleading order equation is,

$$\frac{d^2u^{(1)}}{d\phi^2} + u^{(1)} = 3G_N M\left(u^{(0)}\right)^2,$$

which is like a forced oscillator of unit natural frequency with a periodic forcing source, $3G_N M\left(u^{(0)}\right)^2$. One can check that the solution is,

$$u^{(1)} = \frac{3e\left(G_N M\right)^3}{h^4}\phi\,\sin\left(\phi - \phi_P\right).$$

Thus, to the first post-Newtonian correction, the solution is,

$$u = u^{(0)} + u^{(1)} = \frac{G_N M}{h^2}\left[1 + e\,\cos\left(\phi - \phi_P\right) + 3e\left(\frac{G_N M}{h^2}\right)^2\phi\,\sin\left(\phi - \phi_P\right)\right]$$

$$\approx \frac{G_N M}{h^2}\left[1 + e\,\cos\left(\phi - \phi_P - \psi\right)\right], \quad \psi = 3\left(\frac{G_N M}{h}\right)^2\phi.$$

When the argument of the cosine changes by 2π, i.e.

$$\Delta\left(\phi - \phi_P + \psi\right) = 2\pi$$

one has,

$$\Delta\phi = \frac{2\pi}{1 - 3\left(\frac{G_N M}{h}\right)^2} \approx 2\pi + 6\pi\left(\frac{G_N M}{h}\right)^2$$

i.e. the angle ϕ changes by more than 2π. Thus perihelion itself will move to $\phi_P + 6\pi\left(\frac{G_N M}{h}\right)^2$. Such a motion is called a precession of the perihelion and it is given by the excess,

$$\Delta\phi_P = 6\pi\left(\frac{G_N M}{h}\right)^2$$

per revolution. For an ellipse with semi-major axis, a, one has at the perihelion, the distance from the central mass.

$$r(\phi_P) = a(1-e) = \frac{h^2}{G_N M}\frac{1}{(1+e)}$$

or,

$$h^2 = G_N M a(1-e^2).$$

Using this we get the expression for the perihelion excess of the orbiting planet to be,

$$\Delta\phi_P = \frac{6\pi G_N M}{a\left(1-e^2\right)c^2}.$$

Here the speed of light, c has been restored in the last line to facilitate comparison with observational data. For Mercury, $a = 5.79 \times 10^{10} m$, $e = 0.205$,and hence, $\Delta\phi_P \approx 0.104$ seconds of an arc per revolution. This precession of perihelion of Mercury had been already noticed in 1859, half a century before general relativity was put forward by Einstein. Newtonian theory gave wrong value for the precession of perihelion. When general relativity came along, it provided a very accurate accounting of the "perihelion excess" for Mercury, and is recognized to be one of the four classic tests of general relativity till date.

HW Set 6, Problem : Obtain the vacuum solution to Einstein field equation with a cosmological constant, Λ with spherical symmetry. This solution is called the cosmological Schwarzschild metric. (Hint: The Ricci flat condition gets modified to $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Solve this with the spherically symmetric ansatiz (2))