PH 6458/EP 4258: Gravitation and Cosmology (Fall 2019)

Classic Tests of General Relativity: Gravitational redshift, deflection of light rays & Shapiro time-delay

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The four *classic* experimental tests of general relativity are the gravitational redshift (Pound and Rebka in 1959), (anomalous) perihelion precession (anomaly recognized by 1859, accounted for by Einstein in 1916), deflection of light (Eddington in 1919) and gravitational time delay (computed by Shapiro in 1964, verified by Shapiro at the MIT Lincoln lab in 1966-67). In the previous lecture we have already computed the expression for precession of perihelion of planets by solving the geodesics of massive point particles in the Schwarzschild geometry. Here we obtain the expressions for the three remaining classical tests or effects which apply to light/ EM waves i.e. massless particles.

1 Gravitational time dilation & Gravitational redshift

Consider a static geometry (for example the Schwarzschild) for which the metric is given by,

$$ds^2 = g_{00}(r)\,dt^2 + g_{11}(r)\,dr^2 + r^2 d\Omega_2^2.$$

Consider a process occurring at a fixed location i.e. for which $dr = d\theta = d\phi = 0$ while the duration in the Schwarzschild time coordinate is dt. Then the proper time i.e. time measured by an observer at the location of the process (rest frame), $d\tau(r)$ is,

$$-d\tau^2(r) = g_{00}(r)\,dt^2$$

But then notice that $dt = d\tau(r = \infty) = d\tau_\infty$ i.e. dt is the proper time measured by clocks at spatial infinity, $r = \infty$. So we have,

$$-d\tau^2(r) = g_{00}(r)\,d\tau_\infty^2$$

or,

$$\begin{aligned} d\tau(r) &= \sqrt{-g_{00}(r)}\,d\tau_\infty \\ &= \sqrt{|g_{00}(r)|}\,d\tau_\infty \end{aligned} \tag{1}$$

Since as r decreases, $|g_{00}(r)| = 1 - \frac{2G_N M}{r}$ decreases as well, so clocks tick slower and slower as one moves progressively deeper inside the gravitational field. This phenomenon is called ***gravitational time dilation***. An interesting consequence of this effect is the gravitational redshifting of light as it is emitted from a source (atom) inside the gravitational field and is detected at spatial infinity. For the emission of a single wavelength, we can take, $d\tau(r) = \frac{1}{\nu(r)}$, where $\nu(r)$ is the frequency of the light emitted by the atom as observed in the rest frame of the atom at location, r. Further $d\tau_\infty = \frac{1}{\nu_\infty}$. Plugging these in the gravitational time delay formula we obtain the expression for the ***gravitational redshift*** at spatial infinity

$$\nu_\infty = \nu(r)\sqrt{g_{00}(r)} \tag{2}$$

i.e. $\nu_\infty < \nu(r)$ as $\sqrt{g_{00}(r)} = 1 - \frac{2G_N M}{r} < 1$. For the general case, where the light is emitted and received at different locations inside the gravitational field, say r_e and r_o respectively, one has

$$\frac{\nu_0}{\nu_e} = \sqrt{\frac{g_{00}(r_e)}{g_{00}(r_o)}} \tag{3}$$

If $r_e < r_o$then $\nu_o < \nu_e$ i.e. observed light is redshifted compared to the emitted light, while if $r_o < r_e$, then the observed light is blueshifted compared to the emitted light.

For the Schwarzschild geometry, $|g_{00}(r)| = 1 - \frac{2G_N M}{r}$. Since, $GM/r \ll 1$ one can expand and approximate, $\sqrt{1 - \frac{2G_N M}{r}} \approx 1 - \frac{G_N M}{r}$, and we get the following approximate expression for the redshift

$$\nu_\infty = \nu(r)\sqrt{g_{00}(r)} = \nu(r)\left(1 - \frac{G_N M}{r}\right).$$

This expression for the redshift can also be obtained by treating light to be a point particle with relativistic mass, $m_\nu = E/c^2 = h\nu/c^2$ and conservation of (Newtonian) gravitational potential energy and relativistic energy, $h\nu$

$$\begin{aligned} h\nu_\infty &= h\nu(r) - \frac{GMm_\nu}{r} \\ \Rightarrow \nu_\infty &= \nu(r)\left(1 - \frac{G_N M}{rc^2}\right). \end{aligned}$$

2 Gravitational Deflection of light

Consider propagation of a light ray in the Schwarzschild geometry given by the geodesic equations,

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \tag{4}$$

where λ is a parameter along the geodesic. Since there is no rest frame for light, one cannot choose, $\lambda = \tau$, the "proper time". In addition one has to impose the condition that the geodesic is *null* since it corresponding to light,

$$g_{\mu\nu}dx^\mu dx^\nu = 0 \Rightarrow g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{5}$$

For $\mu = \theta$ component of the geodesic equation, we get the same result as we did for the massive particles case, i.e. $\theta = \frac{\pi}{2}, \forall\lambda$. Thus the entire motion of the light beam is contained on the equatorial plane. The ϕ-component of the geodesic equation gives us a conserved charge, just as it did give the orbital angular momentum for the massive particle case,

$$r^2\frac{d\phi}{d\lambda} = l \Rightarrow \frac{d\phi}{d\lambda} = \frac{l}{r^2}.$$

The t-component of the geodesic equation gives us another conserved charge, analogous to the energy for the massive case,

$$e^{a(r)}\frac{dt}{d\lambda} = \varepsilon.$$

Rechristening, $\lambda \to \frac{\lambda}{\varepsilon}$ and $l \to \varepsilon l$, we have,

$$\frac{d\phi}{d\lambda} = \frac{l}{r^2}, \qquad \frac{dt}{d\lambda} = e^{-a(r)}. \tag{6}$$

Finally we are left with the r-component of the geodesic equation. However, just as we did before in the massless case, we will instead solve the null geodesic condition (5) for the r-component since it is a first order differential equation:

$$-e^{a(r)}\left(\frac{dt}{d\lambda}\right)^2 + e^{b(r)}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \tag{7}$$

or, dividing both sides by $\left(\frac{d\phi}{d\lambda}\right)^2$,

$$-e^{a(r)}\frac{\left(\frac{dt}{d\lambda}\right)^2}{\left(\frac{d\phi}{d\lambda}\right)^2}+e^{b(r)}\left(\frac{dr}{d\phi}\right)^2+r^2=0.$$

Using the charges (6), this equation becomes,

$$-r^4\frac{e^{-a(r)}}{l^2}+e^{b(r)}\left(\frac{dr}{d\phi}\right)^2+r^2=0, \tag{8}$$

which we can solve for $\phi(r)$,

$$\phi(r_P)-\phi(r_Q)=\pm\int_{r_Q}^{r_P}\frac{dr\; e^{b(r)/2}}{\sqrt{r^4\frac{e^{-a(r)}}{l^2}-r^2}} \tag{9}$$

At the distance of nearest approach, call it r_0, one has $\frac{dr}{d\phi}=0$, hence the radial condition gives,

$$-r_0^4\frac{e^{-a(r_0)}}{l^2}+r_0^2=0\Rightarrow l^2=r_0^2\,e^{-a(r_0)}. \tag{10}$$

which we can plug in the solution (9) to obtain,

$$\phi(r_P)-\phi(r_Q)=\pm\int_{r_Q}^{r_P}\frac{dr\; e^{b(r)/2}}{r\sqrt{\frac{r^2}{r_o^2}\frac{e^{a(r_0)}}{e^{a(r)}}-1}}.$$

In particular, for the relevant case of $r_P=r$ and $r_Q=\infty$, i.e. a light ray approaching from infinity, we must choose the upper sign if the light is orbiting counterclockwise and choose the lower sign if the light is orbiting clockwise. Either way,

$$|\phi(r)-\phi(\infty)|=\int_r^{\infty}\frac{dr\; e^{b(r)/2}}{r\sqrt{\frac{r^2}{r_0^2}\frac{e^{a(r_0)}}{e^{a(r)}}-1}} \tag{11}$$

Now recalling the expressions, $e^{a(r)}=1-\frac{2G_NM}{r}$, we can simplify by expanding in the parameters $\frac{G_NM}{r}$ and $\frac{G_NM}{r_0}$

$$\frac{r^2}{r_0^2}\frac{e^{a(r_0)}}{e^{a(r)}}-1=\left(\frac{r^2}{r_0^2}-1\right)\left(1-\frac{2G_NMr}{r_0\,(r+r_0)}+\dots\right). \tag{12}$$

and performing the integral

$$|\phi(r)-\phi(\infty)|=\sin^{-1}\left(\frac{r_0}{r}\right)+\frac{G_NM}{r_0}\left(2-\sqrt{1-\left(\frac{r_0}{r}\right)^2}-\sqrt{\frac{r-r_0}{r+r_0}}\right)+\dots \tag{13}$$

While approaching the central mass, the deflection is of course,

$$|\phi(r_0)-\phi(\infty)|=\frac{\pi}{2}+\frac{2G_NM}{r_0}.$$

While moving away after passing thru the point of nearest approach there will be an identical deflection, so the net angle is,

$$2\,|\phi(r_0)-\phi(\infty)|=\pi+\frac{4G_NM}{r_0}.$$

However if there was no deflection i.e. $M=0$, this gives us π, instead of zero! So the correct deflection formula is obtained by subtracting this π offset,

$$\Delta\phi=2\,|\phi(r_0)-\phi(\infty)|-\pi=\frac{4G_NM}{r_0}. \tag{14}$$

Homework Exercise 1

A. Derive (13) from (11) by first deriving the expression (12) and then integrating it.

B. For the deflection of light by the sun, i.e. $M = M_{\odot}$, the maximum deflection is suffered by light ray which is grazing the sun's surface, i.e. $r_0 = r_{\odot}$. Look up the values of $M_{\odot}, r_{\odot}$ from wikipedia and compute the maximum deflection angle.

3 Shapiro Delay

The final classic test of general relativity has to do with measurement of travel time of light rays instead of deflection angles in the gravitational field of a central object. This is much easier to conduct since it can be done with non-optical light e.g. radar and hence one does not need a special window in time to observe unlike say for deflection of light rays by the sun where one has to wait for total solar eclipses or in the case of precession of perihelion of planets where one has to gather data for centuries. It was proposed by Irwin I. Shapiro in 1964 and carried out by his group at M.I.T's Lincoln laboratory during $1966-67$ by measuring time taken by Radar light signals sent from earth and bounced back off the inner solar planets e.g. Mercury.

To arrive at the expression for the radar time delay, we start from the null metric condition for light rays in the Schwarzschild background, (7)

$$-e^{a(r)}\left(\frac{dt}{d\lambda}\right)^2 + e^{b(r)}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0.$$

Then dividing both sides by $\left(\frac{dt}{d\lambda}\right)^2$ one has,

$$-e^{a(r)} + e^{b(r)}\left(\frac{dr}{dt}\right)^2 + r^2\frac{\left(\frac{d\phi}{d\lambda}\right)^2}{\left(\frac{dt}{d\lambda}\right)^2} = 0.$$

Then substituting the integrals of motion (6), one arrives at,

$$-e^{a(r)} + e^{b(r)}\left(\frac{dr}{dt}\right)^2 + \frac{l^2 e^{2a(r)}}{r^2} = 0,$$

or, using (10),

$$-e^{a(r)} + e^{b(r)}\left(\frac{dr}{dt}\right)^2 + \frac{r_0^2\, e^{-a(r_0)} e^{2a(r)}}{r^2} = 0,$$

which upon integration leads to the expression for the time of passage of light from the point of nearest approach to location r

$$t(r, r_0) = \int_{r_0}^{r} \frac{dr\; e^{\frac{b(r)-a(r)}{2}}}{\sqrt{1 - \frac{r_0^2}{r^2} e^{a(r)-a(r_0)}}}. \tag{15}$$

This is an exact expression (contains all non-linear terms), however for solar system phenomena gravity is sufficiently weak so that only linear order corrections to Newtonian gravity is appreciable. Thus, we will expand the integrand in (15) to linear order in $\frac{G_N M}{r}$ and $\frac{G_N M}{r_0}$, integrating which one gets the expression for the half-time of flight to linear order in $G_N M$,

$$t(r, r_0) \approx \sqrt{r^2 - r_0^2} + 2G_N M \ln\left(\frac{r + \sqrt{r^2 - r_0^2}}{r}\right) + G_N M\sqrt{\frac{r - r_0}{r + r_0}} \tag{16}$$

However recall that we are measuring time on earth where due to gravitational time-dilation, clocks tick slower than the clocks at spatial infinity, (t). So the time observed on earth to linear order in $G_N M$ is,

$$\tau(r, r_0) = \sqrt{-g_{00}(r)}\, t(r, r_0) \approx \sqrt{r^2 - r_0^2} + 2G_N M \ln\left(\frac{r + \sqrt{r^2 - r_0^2}}{r}\right) + G_N M \sqrt{\frac{r - r_0}{r + r_0}} - G_N M \sqrt{1 - \frac{r_0^2}{r}}. \tag{17}$$

Here the leading term $\sqrt{r^2 - r_0^2}$ is of course what we would have obtained if light were to travel in straight line from r to r_0. The other terms which are proportional to $G_N M$ evidently produce a general-relativistic (gravitational) *delay* in the travel of light from r_0 to r.

Homework Exercise 2

A. Derive (16) from (15).

B. For the deflection of light by the mercury at the superior conjunction, the light ray is grazing the sun's surface, i.e. $r_0 = r_\odot$, the sun's radius. For observations on earth, $r = R_\oplus$, the orbital radius of earth. Look up the values of $M_\odot, r_\odot, R_\oplus$ from wikipedia and compute the Shapiro delayed time in μs. (Note speed of light has to be restored in the expression for the Shapiro delayed time).