PH 6458/EP 4258: Gravitation and Cosmology (Fall 2019)

Gravitational Waves

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1 Linearized Einstein Equations

Homework Problem: Consider a decomposition of the metric around Minkowski space,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

Show that upon expanding Einstein field equations and retaining only the terms which are *linear* in $h_{\mu\nu}$ gives the Fierz-Pauli equation,

$$\Box h_{\mu\nu} + (\partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h) - (\partial_\mu\partial^\alpha h_{\alpha\nu} + \partial_\nu\partial^\alpha h_{\alpha\mu}) + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} = -\kappa T_{\mu\nu} \tag{1}$$

and that the metric transformation law,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$$

for a small diffeomorphism,

$$x'^\mu = x^\mu + \Lambda^\mu(x).$$

becomes in the linear limit (linear in both h and in Λ), the familiar spin-2 gauge symmetry,

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x) = h_{\mu\nu}(x) - \partial_\mu\Lambda_\nu(x) - \partial_\nu\Lambda_\mu(x). \tag{2}$$

1.1 "Trace-reversed" field

One can define/introduce a new field, $\bar{h}_{\mu\nu}$ by,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{3}$$

and rewrite the Fierz-Pauli equation in these variables. This simplifies/shortens the equation a little bit,

$$\Box\bar{h}_{\mu\nu} - \left(\partial_\mu\partial^\alpha\bar{h}_{\alpha\nu} + \partial_\nu\partial^\alpha\bar{h}_{\alpha\mu}\right) + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} = -\kappa T_{\mu\nu}. \tag{4}$$

However, the gauge transformation (2) now reads,

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu + \eta_{\mu\nu}\left(\partial.\Lambda\right). \tag{5}$$

Sean Carroll mentions in his book/notes that this barred field is sometimes called as the "*trace-reversed*" gravitational field (potential) because, its trace, $\bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu}$ according to the definition

$$\bar{h} = -\left(\frac{D}{2} - 1\right) h$$

i.e. has the opposite sign to that of the Fierz-Pauli field, $h_{\mu\nu}$.

2 Gravitational Waves

2.1 Warm up example: EM waves

Again by way of example we recall the wave solutions in electromagnetism. There, one can use the U(1) gauge freedom, to choose a gauge condition. We use this to get to the Lorenz gauge condition,

$$\partial_\mu A^\mu = 0, \tag{6}$$

which simplifies the field equation in the absence of sources, $j = 0$ to the wave equation with wave-velocity c,

$$\partial^2 A^\mu = 0.$$

However we will see in the following that Lorenz gauge is only a partial gauge fix. To see this we conduct a gauge transformation by a function, say χ i.e.

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi \tag{7}$$

we see that gauged transformed potential, A'^μ has a divergence

$$\partial_\mu A'^\mu = \underbrace{\partial_\mu A^\mu}_{=0} + \partial^2 \chi = \partial^2 \chi.$$

Thus if $\partial^2 \chi = 0$, then even after the gauge transformation, the new or transformed field A'^μ still satisfies Lorenz gauge condition, $\partial_\mu A'^\mu = 0$. So the Lorenz gauge does not uniquely fix the gauge and we have a residual degree of freedom, χ which is a harmonic function. So the total number of unrestrained/independent degrees of the EM field is,

$$\underbrace{D}_{\text{d.o.f. of } A^\mu} - \underbrace{1}_{\text{Lorenz gauge}} - \underbrace{1}_{\text{Residual gauge}} = D - 2.$$

Now let's look at plane wave solutions. If we substitute a plane wave solution as an ansatz,

$$A^\mu = \epsilon^\mu \exp(i k_\alpha x^\alpha), \tag{8}$$

the equation of motion, namely, $\partial^2 A^\mu = 0$ implies,

$$\begin{aligned} \partial^2 \left[\epsilon^\mu \exp\left(i k_\alpha x^\alpha\right)\right] = 0, \\ \implies \epsilon^\mu \partial^2 \left[\exp\left(i k_\alpha x^\alpha\right)\right] = 0, \\ \implies -\epsilon^\mu k^2 \exp\left(i k_\alpha x^\alpha\right) = 0. \end{aligned}$$

This has to be true for arbitrary x, and since $\epsilon^\mu \neq 0$, the only solution is $k^2 \equiv k^\alpha k_\alpha = 0 \implies k^0 = \pm|\mathbf{k}|$. Recall that the by definition, $k^0 = \omega/c$, thus we have,

$$\omega = c|\mathbf{k}|.$$

Thus the waves travel will the invariant signal speed, c. Further, plugging the plane wave ansatz (8) the Lorentz gauge condition (6) gives,

$$k_\mu \epsilon^\mu = 0. \tag{9}$$

This implies ϵ^μcannot be arbitrary, they must satisfy momentum space Lorentz gauge constraint (9). To see which polarizations are allowed, we consider the wave to be moving along z-axis, i.e. $k = k\,\hat{\mathbf{e}}_3$. Then the wave equation forces, $k^0 = \pm k$. Let's work with the upper sign, i.e. $k^\mu = (ck, 0, 0, k)$, which represents planar fronts propagating in the direction of the positive z-axis. Then the Lorentz gauge condition (9) gives,

$$-\epsilon^0 + \epsilon^1 = 0.$$

Thus we can have three independent solutions,

$$\epsilon_{(1)} = (0, 1, 0, 0), \epsilon_{(2)} = (0, 0, 1, 0), \epsilon_{(3)} = (1, 0, 0, 1).$$

These are the three polarization vectors, namely the transverse polarizations, $\epsilon_{(1)}, \epsilon_{(2)}$ and longitudinal polarization, $\epsilon_{(3)}$. We call it longitudinal since it is parallel to the wave vector, $\epsilon^\mu_{(3)} = \omega k^\mu$. **This longitudinal polarization, $\epsilon_{(3)}$ gives, $F^{\mu\nu} = 0$ and hence can be omitted (Check this**). Thus we recover the result that there are only 2 d.o.f. in the EM wave.

We can generalize the discussion to D spacetime dimensions and wave propagation in arbitrary direction, say $\mathbf{k}$. In such case the wave vector is, $k^\mu = \big(k^0 = c|\mathbf{k}|, \mathbf{k}\big)$consistent with the solution to the wave equation, $k_\mu k^\mu = 0$ then gives, $\omega = \pm k$ and again we take the upper sign. Then *one subset* of the polarization vectors, ϵ^μ which satisfy Lorenz gauge, i.e. $\epsilon^\mu k_\mu = 0$ can be taken to be:

$$\epsilon^\mu_\perp = (0, \boldsymbol{\epsilon})\,,\ \boldsymbol{\epsilon} \cdot \mathbf{k} = 0. \tag{10}$$

Given/for a fixed $\mathbf{k}$, this condition says that any spatial vector, $\boldsymbol{\epsilon}$ which is perpendicular to $\mathbf{k}$ is an allowed solution. Since there are $(D-2)$ directions perpendicular to $\mathbf{k}$-vector, there are $(D-2)$ such independent polarizations each of which are spatially orthogonal to the direction of the propagation of the wave and these can be chosen to be perpendicular to each other as well. We will denote these by an index, i:

$$\epsilon^\mu_i = (0, \boldsymbol{\epsilon}_i)\,, i = 1, \ldots, (D-2). \tag{11}$$

These are, for obvious reasons, called the ***transverse*** polarizations. However, in addition to the transverse polarizations, there is one more solution, say $\epsilon^\mu_{(D-1)}$, to the constraint (9) - one which is parallel/proportional to the wave-vector,

$$\epsilon^\mu_{(D-1)} = \frac{1}{\omega} k^\mu.$$

This also satisfies Lorentz gauge, $\epsilon^\mu_{(D-1)} k_\mu = 0$ since, k^μ itself is a null vector. Thus we have constructed/exhausted all possible independent plane wave polarizations. However we know that Lorenz gauge does not eliminate all redundant degrees of freedom i.e. it is only a partial gauge fix. The residual gauge freedom within Lorenz gauge is a gauge transformation by a harmonic function, say χ. Now any harmonic function i.e. one which satisfies $\partial^2\chi = 0$, can be expressed in the form of plain waves again

$$\chi(x) = \tilde{\chi} e^{ik.x}, k^2 = 0,$$

with a constant, $\tilde{\chi}$. After doing this residual gauge transformation by the function, χ the polarization vector, ϵ^μ gets gauge-transformed to,

$$\epsilon^\mu \to \epsilon'^\mu = \epsilon^\mu + ik^\mu\tilde{\chi}.$$

In particular, we can choose$\tilde{\chi} = i/\omega$. That means,

$$\epsilon^{\mu} \to \epsilon'^{\mu} = \epsilon^{\mu} - \frac{1}{\omega} k^{\mu}.$$

In particular, the polarization, $\epsilon^{\mu}_{(D-1)}$ gets killed since it is equal to k^{μ}/ω,

$$\epsilon^{\mu}_{(D-1)} \to \epsilon'^{\mu}_{(D-1)} = 0.$$

Polarizations such as $\epsilon^{\mu}_{(D-1)}$ which can be by gauge transformed zero (no field) solution is called a ***pure gauge***, since it is not a physical solution but an artifact of incomplete gauge fixing.

2.2 Gravitational Waves

For the case of gravity we can also adopt such a Lorentz covariant gauge condition, called the Hilbert gauge or de Donder gauge or Fock gauge or Einstein gauge or simply the *Harmonic* gauge,

$$\partial^{\mu} \bar{h}_{\mu\nu} = \partial_{\mu} \left(h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \right) = 0. \tag{12}$$

then the Equation of motion (4) becomes the wave equation,

$$\partial^2 \bar{h}_{\mu\nu} = 0. \tag{13}$$

However, just like the electromagnetic case, this gauge condition does not fix the field completely and there remains a class of residual gauge transformations. Let's say that we have chosen our field, $\bar{h}_{\mu\nu}$ to satisfy the de Donder gauge. Now if we further gauge transform by an arbitrary vector field $\chi^{\mu}(x)$, the gauge transformed field is given by (cf (5)),

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_{\mu}\chi_{\nu} - \partial_{\nu}\chi_{\mu} + \eta_{\mu\nu} \left(\partial . \chi\right),$$

while the four-divergence of this new/gauge-transformed field is,

$$\begin{aligned} \partial^{\mu} \bar{h}'_{\mu\nu} &= \underbrace{\partial^{\mu} \bar{h}_{\mu\nu}}_{=0} - \partial^{\mu}\partial_{\mu}\chi_{\nu} - \cancel{\partial^{\mu}\partial_{\nu}\chi_{\nu}} + \cancel{\eta_{\mu\nu}\partial^{\mu} \left(\partial . \chi\right)} \\ &= -\partial^2 \chi_{\nu}. \end{aligned}$$

Now, if the components of the vector field, χ_{μ} are harmonic functions, then, $\partial^{\mu} \bar{h}'_{\mu\nu} = 0$ as well i.e. is also satisfies de Donder gauge. Thus even within the Harmonic gauge, we have the residual gauge freedom choose D number of harmonic functions (the D components of χ_{μ}). Once we fix, χ_{μ}we have fixed the gauge completely. Therefore, the total number of independent degrees of freedom of the gravitational field is,

$$\underbrace{\frac{D(D+1)}{2}}_{\text{d.o.f of symmetric tensor}} - \underbrace{D}_{\text{Lorenz gauge conditions}} - \underbrace{D}_{\text{Residual gauge conditions}} = \frac{D(D-3)}{2}.$$

In 3+1 dimensions, $D = 4$, hence, the number of degrees of freedom is, $4(4-3)/2 = 2$. To completely fix the gauge, we choose the a χ_{μ}so that the gauge transformed field, $\bar{h}'_{\mu\nu}$ satisfies,

$$\bar{h}' \equiv \eta^{\mu\nu} \bar{h}'_{\mu\nu} = 0, \quad \bar{h}'_{0i} = 0. \tag{14}$$

Here the index i only runs over the spatial directions. This sub gauge of the de Donder gauge is called the **transverse traceless gauge.**

Next, we look for plane wave solutions in the transverse traceless gauge. We plug in the plane-wave ansatz, namely,

$$\bar{h}_{\mu\nu} = \epsilon_{\mu\nu}\, e^{ik_\mu x^\mu}, \tag{15}$$

in the in the field equations (13), and get the condition,

$$k^2 \equiv k^\mu k_\mu = 0, \tag{16}$$

i.e. the wave vector is a null vector, and hence gravitational waves, just like EM waves, propagate with the invariant speed, c. The symmetric tensor $\epsilon^{\mu\nu}$is called **Polarization tensor**. The gauge condition (12) gives us the restriction on the polarization tensor,

$$\epsilon^{\mu\nu} k_\mu = 0. \tag{17}$$

Under the residual gauge transformations (i.e. by a vector field, χ_μ, whose components are harmonic gauge), i.e.

$$\chi_\mu = \tilde{\chi}_\mu e^{ik^\alpha x_\alpha},$$

where $\tilde{\chi}_\mu$ is independent of the position and $k^\alpha$is the gravitational wave vector again, the polarization vector transforms to,

$$\epsilon'_{\mu\nu} = \epsilon_{\mu\nu} - ik_\mu \tilde{\chi}_\nu - ik_\nu \tilde{\chi}_\mu + i\,\eta_{\mu\nu} k.\tilde{\chi}. \tag{18}$$

To explicitly construct the polarization tensor in transverse traceless gauge in general spacetime dimensions, D, we make use of the transverse polarization vectors, Eq. (11), of the EM wave One can take symmetric bilinear products of these EM wave transverse polarization tensor, and the wave vector to write down a basis of candidate polarization tensors for the gravitational waves that satisfy harmonic gauge:

$$\epsilon^{(ij)}_{\mu\nu} = \epsilon_{(i)\mu}\epsilon_{(j)\nu} + \epsilon_{(i)\nu}\epsilon_{(j)\mu}, \quad i \neq j \tag{19}$$

$$\epsilon^{(ii)}_{\mu\nu} = \epsilon_{(i)\mu}\epsilon_{(i)\nu}. \tag{20}$$

$$\epsilon^{(i0)}_{\mu\nu} = \epsilon_{(i)\mu} k_\nu + \epsilon_{(i)\nu} k_\mu, \tag{21}$$

and

$$\epsilon^{(00)}_{\mu\nu} = k_\mu k_\nu. \tag{22}$$

Here as before i, j run over the transverse directions, $i = 1, \ldots, (D-2)$. These satisfy Harmonic gauge because the EM polarization vectors, as well as the wave-vector itself satisfy Lorenz gauge. However both $\epsilon^{\mu\nu}_{(i0)}$ and $\epsilon^{\mu\nu}_{(00)}$ can be shown to be "pure gauge", i.e. can be removed by doing a residual gauge transformation (18) to $\epsilon^{\mu\nu} = 0$. To show $\epsilon^{\mu\nu}_{(i0)}$ is pure gauge, one needs to choose,

$$\tilde{\chi}_\mu = -i\epsilon_{(i)\mu},$$

while to show $\epsilon^{\mu\nu}_{(00)}$ is pure gauge, one needs to choose,

$$\tilde{\chi}_\mu = -\frac{i}{2} k_\mu.$$

Thus, we are left with just the polarizations, $\epsilon^{(ii)}$ and $\epsilon^{(ij)}$ defined through equations (20) and (19). There are $\binom{D-2}{2} + D - 2 = \frac{(D-2)(D-1)}{2}$ number of these polarization tensors. Let's check if these polarization tensors satisfy the transverse traceless subgauge conditions. Note that since the transverse polarizations of EM waves have no time-component, $\epsilon_i^0 = 0$, these automatically satisfy the subgauge condition $\epsilon_{0\nu} = 0$. The Polarization tensors, $\epsilon_{(ij)}^{\mu\nu}$ are definitely traceless, but the $\epsilon_{(ii)}^{\mu\nu}$ are not. However it is easy to construct traceless combinations,

$$\epsilon_{(ii)}^{\mu\nu} \to (D-2)\,\epsilon_{(i)}^{\mu}\epsilon_{(i)}^{\nu} - \sum_{j=1}^{D-2} \epsilon_{(j)}^{\mu}\epsilon_{(j)}^{\nu}. \tag{23}$$

This tracelessness reduces the number of independent $\epsilon_{(ii)}^{\mu\nu}$'s by one as the last one can be expressed in terms of the others,

$$\epsilon_{(D-2,D-2)}^{\mu\nu} = 1 - \sum_{j=1}^{D-3} \epsilon_{(j}^{\mu}\epsilon_{(j)}^{\nu}.$$

So the total number of independent polarization tensors which are transverse and traceless, given by $\epsilon_{(ij)}^{\mu\nu}$ of equation (19) and by $\epsilon_{(ii)}^{\mu\nu}$ as in equation (23), are

$$\binom{D-2}{2} + D - 3 = \frac{(D-2)\,(D-3)}{2} + D - 3 = \frac{D\,(D-3)}{2}.$$

As an example, let's do the $D = 4$ i.e. 3 +1 dimensional spacetime. Here, $D - 2 = 2$ so the transverse indices, $i, j = 1, 2$. Let's choose the direction of propagation of the wave to be z-axis, then wave vector is, $k^\mu = (k, 0, 0, k)$. Then EM wave transverse polarizations for this case an be chosen to be,

$$\epsilon_{(1)}^{\mu} = (0, 1, 0, 0)\,,\ \epsilon_{(2)}^{\mu} = (0, 0, 1, 0)$$

Then two independent polarization tensors for the gravitational wave are (in matrix form)

$$\epsilon_{\mu\nu}^{(12)} = \epsilon_{(1)\mu}\epsilon_{(2)\nu} + \epsilon_{(2)\mu}\epsilon_{(1)\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\epsilon_{\mu\nu}^{(11)} = 2\epsilon_{(1)\mu}\epsilon_{(1)\nu} - \sum_{j=1}^{2} \epsilon_{(j)\mu}\epsilon_{(j)\nu} = \epsilon_{(1)\mu}\epsilon_{(1)\nu} - \epsilon_{(2)\mu}\epsilon_{(2)\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Homework Problem: Write down the independent gravitational polarization tensors in in $D = 5$ spacetime dimensions i.e. 4 space and 1 time using the same method as above.

2.3 "+" and "×" type polarizations of gravitational waves

In the literature for $D = 4$, the $\epsilon^{(11)}$ polarization of the gravitational wave is known as the + type, denoted by $\epsilon^{\oplus}$, while the $\epsilon^{(12)}$ type polarization is referred to as the $\epsilon^{\otimes}$ ("cross"). The point of this nomenclature is as explained in what follows. Consider a gravitational wave incident on a bunch of point particles at rest on the 12-plane i.e. xy.-plane. Their relative motion as a result of this gravitational wave disturbance incident on them is given by the geodesic deviation equation,

$$\frac{D^2 S^\mu}{D\tau^2} = R^\mu{}_{\sigma\nu\rho} T^\sigma T^\nu S^\rho, \frac{D^2}{D\tau^2} = (T.\nabla)^2 .$$

In the 12 plane, the separation vector is, $S^\mu = \left(0, \frac{\Delta x^1}{\Delta s}, \frac{\Delta x^2}{\Delta s}, 0\right)$. Now for particles at rest and in general for slow moving particles, $T^\nu = \frac{dx^\nu}{d\tau} = (\gamma, \gamma \mathbf{v}) \approx (1,0,0,0)$. Also in the slow approximation, $\frac{D}{D\tau} \approx \frac{\partial}{\partial \tau} \approx \frac{\partial}{\partial t}$. Thus, the geodesic deviation equation becomes,

$$\frac{\partial^2 S^\mu}{\partial t^2} = R^\mu{}_{00\rho} S^\rho.$$

The linearized Riemann tensor is,

$$R^\mu{}_{00\rho} = \frac{1}{2}\left(\partial_0^2 h^\mu_\rho - \partial_\rho \partial_0 h^\mu_0 + \partial^\mu \partial_\rho h_{00} - \partial_0 \partial^\mu h_{0\rho}\right).$$

Let's consider the case when $\mu = 1$, i.e.,

$$\frac{\partial^2 S^1}{\partial t^2} = R^1{}_{00\rho} S^\rho.$$

Now,

$$R^1{}_{00\rho} = \frac{1}{2}\left(\partial_0^2 h^1_\rho - \partial_\rho \partial_0 h^1_0 + \partial^1 \partial_\rho h_{00} - \partial_0 \partial^1 h_{0\rho}\right)$$

In transverse traceless case, $h^1_0 = 0$. Also $\rho = 1, 2$ for S being in the 12-plane. Then,

$$R^1{}_{001} = \frac{1}{2}\left(\partial_0^2 h^1_1 - \partial_1 \partial_0 h^1_0 + \partial^1 \partial_1 h_{00} - \partial_0 \partial^1 h_{01}\right)$$

$$R^1{}_{002} = \frac{1}{2}\left(\partial_0^2 h^1_2 - \partial_2 \partial_0 h^1_0 + \partial^1 \partial_2 h_{00} - \partial_0 \partial^1 h_{02}\right)$$

For the $\epsilon^{(11)}$ polarization, $h^1_1 = C_+ e^{ik.x}$, and $R^1{}_{001} = \frac{1}{2}\partial_0^2 h^1_1$, $R^1{}_{002} = 0$, hence the geodesic deviation equation becomes,

$$\frac{\partial^2 S^1}{\partial t^2} = R^1{}_{001} S^1$$

$$\begin{aligned}\frac{\partial^2 S^1}{\partial t^2} &= R^1{}_{001} S^1 \\ &= \frac{1}{2}\partial_0^2 h^1_1 S^1 \\ &= \frac{1}{2}\frac{\partial}{\partial t^2}\left(C e^{ik.x}\right) S^1.\end{aligned}$$

Similarly noting that for the $\epsilon^{(11)}$ polarization, $h_2^2 = -C_+ e^{ik.x}$, one can show,

$$\begin{aligned}\frac{\partial^2 S^2}{\partial t^2} &= \frac{1}{2}\partial_0^2 h_2^2 S^2. \\ &= \frac{1}{2}\frac{\partial}{\partial t^2}\left(-C e^{ik.x}\right) S^2.\end{aligned}$$

The solution to these equations to lowest order are,

$$\begin{aligned}S^1(t) &= \left(1 + \frac{C_+}{2} e^{ik.x}\right) S^1(0), \\ S^2(t) &= \left(1 - \frac{C_+}{2} e^{ik.x}\right) S^2(0).\end{aligned}$$

Thus, particles initially separated in the x^1 or x direction will oscillate back and forth in the x direction, and while those with an initially separation along y direction with oscillate in the y-direction. That is, if we start with a circular ring of stationary particles in the xy plane, as the wave passes they will bounce back and forth in the shape of a "+".

For the $\epsilon^{(12)}$ polarized gravitational waves, $h_{12} = h_{21} = C_\times e^{ik.x}$, and the geodesic deviation equations lead to,

$$\begin{aligned}S^1(t) &= S^1(0) + \frac{C_\times}{2} e^{ik.x} S^2(0), \\ S^2(t) &= S^2(0) + \frac{C_\times}{2} e^{ik.x} S^1(0).\end{aligned}$$

That is, if we start with a circular ring of stationary particles in the xy plane, as the wave passes they will bounce back and forth in the shape of a "×".