PH 6548/4258: Cosmology Lecture 3

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1 Cosmic Dynamics: Friedmann and Raychaudhuri Equation

Cosmology treats the universe as a whole as a (Lorentzian) manifold and the evolution/ dynamics of the universe is governed by general relativity i.e. the Einstein field equations. As such, the starting point is cosmology is an expression for the metric describing the universe. On the basis of the ***cosmological principle***, i.e. purely symmetries alone, the metric for the universe can be narrowed down to the Robertson-Walker form

$$ds^2 = -dt^2 + g_{ij}\, dx^i\, dx^j, \tag{1}$$

where the g_{ij} is the metric of the 3-dimensional spatial sections parameterized by x^i's. Further the metric for the spatial sections, has time and space dependencies factored,

$$g_{ij} = a^2(t)\, \gamma_{ij}\left(\{x_i\}\right)$$

where in various different coordinate systems,

$$\gamma_{ij} dx^i dx^j = \begin{cases} d\rho^2 + f_\kappa^2(\rho) d\Omega_2^2, \{x^i\} = \rho, \theta, \phi \\ \frac{dr^2}{1-\kappa\frac{r^2}{R_0^2}} + r^2 d\Omega_2^2, \{x^i\} = r, \theta, \phi \\ \delta_{ij} + \frac{\kappa}{R_0^2}\frac{x_i x_j}{1-\frac{\kappa}{R_0^2}\mathbf{x}\cdot\mathbf{x}}, \{x^i\} = x, y, z \end{cases}. \tag{2}$$

The dynamics/evolution of the universe is given by the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N\, T^M_{\mu\nu}.$$

One can move the cosmological constant term to RHS and think of it as a component of the stress tensor arising from empty spacetime/vacuum itself, called *vacuum energy*,

$$T^\Lambda_{\mu\nu} = -\frac{\Lambda}{8\pi G_N} g_{\mu\nu}. \tag{3}$$

and rewrite the field equations as,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N\, T_{\mu\nu}, \tag{4}$$

where the stress tensor now is the *total* stress tensor of matter as well as vacuum, $T_{\mu\nu} = T^M_{\mu\nu} + T^\Lambda_{\mu\nu}$.

Homework 7.2: Christoffel Symbols, Ricci tensor for FLRW metric

A. Show that for the FLRW metric, the nonvanishing Christoffel symbols are,

$$\Gamma^0_{ij} = \frac{\dot{a}}{a} g_{ij}, \qquad \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j, \qquad \Gamma^i_{jk} = \Gamma^{(3)i}_{jk}$$

where $\Gamma^{(3)i}_{jk}$ are the Christoffel symbols derived from the 3-metric, γ_{ij}..

B. Show that the components of the Ricci tensor and are,

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad R_{0i} = 0, \qquad R_{ij} = \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{\kappa}{a^2 R_0^2}\right) g_{ij}$$

You may use either of the different 3-metric parameterizations (2) or you may carry this out for the generic case. If you use the Mathematica code provided, just attach a print out of the Mathematica output.

From the above exercise one can immediately extract the components of the Einstein tensor for the FLRW geometry,

$$G_{00} = 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3\kappa}{a^2 R_0^2}, \qquad G_{0i} = 0, \qquad G_{ij} = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2 R_0^2}\right).$$

and thus the LHS of the Einstein field equations is readily evaluated. However in the RHS we have the stress tensor whose form we are still to determine. As a simplifying assumption we will assume the matter to be of the perfect fluid form[1],

$$T_{\mu\nu} = (\rho + P)\, u_\mu u_\nu + P\, g_{\mu\nu}$$

where recall that ρ, P are the density and pressure in the fluid rest frame which is same as the cosmic frame (1). In the rest frame, it takes the simple form,

$$T_{00} = \rho, \qquad T_{ij} = P\, g_{ij}, \qquad T_{0i} = 0$$

Spatial homogeneity and isotropy imply that ρ and P cannot depend on position or direction, i.e no dependence on x_i's. Thus, $\rho = \rho(t)$ and $P = P(t)$. Plugging in the fluid form ansatz for the stress tensor in the RHS of 00-component of the Einstein field equation (4) then give us,

$$\begin{aligned} G_{00} &= 8\pi G_N T_{00} \\ \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G_N}{3}\rho - \frac{\kappa}{a^2 R_0^2} \end{aligned} \tag{5}$$

This equation is known as the ***Friedmann Equation*** in cosmology.

[1] This assumption will be justified by the remarkably spot on predictions from the equations that follow on the basis of this assumption.

The ij-component of the Einstein field equations (4) on the other hand give us,

$$G_{ij} = 8\pi G_N T_{ij}$$
$$\Rightarrow 2\frac{\ddot{a}}{a} = -8\pi G_N P - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2 R_0^2}$$

which when combined with the Friedmann equation (5) leads to,

$$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\left(\rho + 3P\right). \tag{6}$$

The equation is referred to as the ***Raychaudhuri equation*** (1955) or the ***acceleration equation*** in cosmology.

Finally there is the covariant conservation law for the stress tensor, $\nabla_\mu T^{\mu\nu} = 0$. For $\nu = 0$, this leads to the evolution equation for energy density,

$$\dot{\rho} + 3\,\frac{\dot{a}}{a}\,(\rho + P) = 0. \tag{7}$$

For $\nu = i$, the covariant conservation of the stress tensor does not lead to any equation as the LHS vanishes identically.

Homework 7.3: Spatial component of covariant continuity of stress tensor

Show that for $\nu = i$, the covariant continuity equation for the stress tensor, $\nabla_\mu T^{\mu\nu} = 0$ is automatically satisfied as the LHS vanishes identically.

At this point, one might feel pleased that to solve for three unknown functions $a(t), \rho(t), P(t)$, we have at our disposal three equations, Friedmann (5), Raychaudhuri (6) and continuity (7), but the fact is that these three equations are not independent. One can easily verify that taking a time derivative of both sides of the Friedmann equation and then substituting for $\dot{\rho}$, the expression obtained from the continuity equation, one just gets back the Raychaudhuri equation! So in reality, one has two equations in three variables and unless some extra information is provided, these equations cannot be solved for three variables. One such extra piece of information is the ***equation of state*****,** which expresses, the pressure as a function of the density,

$$P = P(\rho).$$

In general the pressure is a complicated function of the energy density, ρ. However since most of the matter in the universe is in such a dilute state (recall the energy density of the universe is few Hydrogen atoms per cubic meters), one can Taylor expand $P(\rho)$ around $\rho = 0$, and keep on the leading order linear term,

$$P = w\,\rho.$$

This coefficient, w is called the equation of state parameter. To get some feel for this parameter, w we consider some common examples of sources of matter energy in the universe. Let's begin with non-relativistic matter, which is mostly in a dilute gaseous form. For such dilute gases, the equation of state is described by the Ideal gas law,

$$PV = N\,k\,T$$

or, equivalently, after introducing μ, the mass of the gas molecules/atoms/constituents,

$$P = \frac{k\,T}{\mu}\left(\underbrace{\frac{N\,\mu}{V}}_{\rho}\right)$$
$$= \frac{k\,T}{\mu}\rho.$$

Since the gas is non-relativistic (cold), we are justified in replacing the rest mass-density by the energy density, the kinetic energy can be ignored. Also, for an ideal gas in three dimensions, one has from the equipartition theorem,

$$\frac{1}{2}m\langle\mathbf{v}^2\rangle = \frac{3}{2}k\,T.$$

Substituting this in the gas law, we get,

$$P = \frac{1}{3}\langle\mathbf{v}^2\rangle\,\rho.$$

Again for non-relativistic particles, $|\mathbf{v}| \ll 1$, to a good approximation (linear order in $|\mathbf{v}|$),

$$P = 0!$$

i.e. pressure of the non-relativistic gas vanishes. Such a pressureless source of matter-energy is dubbed ***dust***. One can check that up to temperatures of billions of Kelvin, electrons and to trillions of Kelvin, protons remain non-relativistic. So most of the "cold" matter in the universe can be treated as dust. Thus we have established that the equation of state parameter for matter/dust is $w_m = 0$.

Now one might wonder what about very hot gas or neutrinos, for which the velocities of the molecules are close to speed of light, or massless particles such as photons or gluons or gravitons for which the speed of particles is exactly the speed of light. What is the equation of state parameter for them? For a gas of photons/EM radiation gas, we have the thermodynamic relation,

$$P = \frac{1}{3}\rho.$$

Thus for photon gas, $w = \frac{1}{3}$. For ultrarelativistic particles we have $\langle\mathbf{v}^2\rangle \approx 1$, and hence their pressure,

$$P = \frac{1}{3}\langle\mathbf{v}^2\rangle\,\rho = \frac{1}{3}\rho.$$

So for both photons and ultrarelativistic matter (hot matter), one has, $w_\gamma = \frac{1}{3}$. So to simplify terminology, we will dub gas of massless and gas of ultrarelativistic (hot) particles alike as ***radiation***.

According to the Raychaudhuri equation (6), for a arbitrary equation of state parameter, w, one has $\ddot{a} \propto -(1+3w)$. Thus, for both dust and radiation, $\ddot{a} < 0$, i.e. leads to a decelerating universe, the expansion of the universe slows down with time. One the other hand for, $w < -\frac{1}{3}$, $\ddot{a} > 0$, i.e. universe whose expansion rate is picking up! Such a strange source of matter-energy is called ***Dark Energy***, a term coined by the cosmologist, Michael Turner.

Source	w
Matter (dust)	0
Radiation	$\frac{1}{3}$
Vacuum Energy/Cosmological constant	-1
Dark Energy	$< -\frac{1}{3}$
Curvature	$-\frac{1}{3}$

Table 1: Equation of state parameters for diverse sources of matter-energy in the universe

One can ask what is the equation of state parameter for vacuum energy (cosmological constant). Vacuum energy is described by the stress tensor form, (3), which implies $\rho_\Lambda = \frac{\Lambda}{8\pi G_N}$, and $P_\Lambda = -\frac{\Lambda}{8\pi G_N}$. So one has, $\omega_\Lambda = -1$! Thus vacuum energy or a cosmological constant qualifies to be a candidate for dark energy!

Finally, sometimes one even considers the term in the Friedmann equation arising from the curvature of spatial sections, i.e. $-\frac{\kappa}{a^2R^2}$, and dresses it up as a matter-energy source, with $\rho_\kappa = -\frac{3\kappa}{8\pi G_N a^2 R_0^2}$. Then according to Raychaudhuri equation one must set the pressure due to curvature, $P_\kappa = -\frac{1}{3}\rho_\kappa$. So the equation of state parameter for curvature is, $w_\kappa = -\frac{1}{3}$, and does not qualify to be a dark energy candidate. The equation of state parameters of various components of matter-energy in the universe are summarized in the table (1).

One might get an impression that perhaps the equation of state parameter can assume arbitrary values, positive or negative. However an important restriction comes from the requirement that the speed of sound i.e. pressure waves (adiabatic), $v_s = \frac{\partial P}{\partial \rho}$, must be less than the speed of light, i.e.,

$$w < 1.$$

1.1 Critical density and Spatial curvature

The Friedmann equation, (5) can be recast in a more elegant dimensionless form, when one divides both sides by the square of the Hubble parameter, $H^2 = \left(\frac{\dot{a}}{a}\right)^2$, and then rearranging the terms a bit,

$$\frac{\rho}{\frac{3H^2}{8\pi G_N}} - 1 = \frac{\kappa}{a^2H^2R_0^2}$$

The quantity on the denominator in the LHS, $\frac{3H^2}{8\pi G_N}$ has dimensions of mass (energy) density and is called the ***critical energy density*** of the universe and labeled as ρ_c. , Further defining the ***density paramter***, as the ratio, $\Omega = \frac{\rho}{\rho_c}$, the Friedmann equations acquires a cute form,

$$\Omega - 1 = \frac{\kappa}{a^2H^2R_0^2}.$$

This form of the Friedmann equation has immediate physical implications since the sign of the RHS is totally determined by the sign of the spatial curvature i.e. κ, the rest are square quantities and are positive. This means one must have

$$\Omega > 1 \Rightarrow \kappa = +1,$$

$$\Omega < 1 \Rightarrow \kappa = -1,$$

$$\Omega = 0 \Rightarrow \kappa = 0.$$

i.e. a universe containing matter-energy sources with density exceeding the critical density (overdense universe) must have positive curvature sections, a universe with matter-energy density less than the critical density (underdense universe) must have negative curvature spatial sections and for matter-energy density equal to critical density (critical universe) one must have flat (zero curvature) spatial sections! Since the spatial curvature is a fixed once and for all, an overdense universe is destined to remain overdense forever, an underdense/subcritical universe is destined to remain underdense forever and a critical universe remains critical forever. Let's estimate the critical density for the present day universe using the value of the present day Hubble parameter, $H(t_0)$, aka the Hubble constant, $H_0 = 70$ km/s/Mpc. Thus,

$$\rho_c(t_o) = \frac{3H_0^2}{8\pi G_N} \approx 10^{-26}\, kg/m^3.$$

Alternatively, in astronomical units,

$$\rho_c(t_0) \approx 140\,\mathrm{M}_{\odot}/kpc^3,$$

and in particle physics units,

$$\rho_c(t_0) \approx 5.2\, GeV/m^3.$$

This is approximately around 5 protons/ Hydrogen atoms per cubic meter. So the critically dense universe is surprisingly dilute!

1.2 An aside on the vacuum energy or cosmological constant: Einstein Static Universe

Einstein, in 1917, was the first to try to construct a model for the universe. Although the CMB was not yet detected, he correctly concluded the matter (dust) is far more abundant in the universe. Assuming that the matter-energy in the universe is entirely dust, one has

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho_m - \frac{\kappa}{a^2 R_0^2},$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\rho_m.$$

However, since the galactic redshift phenomenon was not yet discovered, from simply looking at the motion of the stars inside the milky way (which is decoupled from the Hubble expansion), Einstein incorrectly presumed that the universe is static or unchanging, i.e. $\dot{a}, \ddot{a}, \dot{\rho} = 0$. The Raychaudhuri equation is incompatible with $\ddot{a} = 0$ unless, $\rho_m = 0$, i.e. an empty universe! So to obtain a static universe, he introduced the cosmological constant in the Einstein field equations. In the presence of the cosmological constant, the equations governing the scale factor evolution change,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho_m - \frac{\kappa}{a^2 R_0^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\rho_m + \frac{\Lambda}{3}.$$

Demanding $\ddot{a} = 0$ now determines the value of the cosmological constant,

$$\Lambda = 4\pi G_N \rho_m,$$

which is necessarily positive. Then demanding $\dot{a} = 0$ then gives, $\kappa = 1$, and

$$R_0 = \frac{1}{\sqrt{4\pi G_N \rho_m}} = \frac{1}{\sqrt{\Lambda}}.$$

After Hubble's discovery of the galactic redshifts, he along with the most of the physics community abandoned the cosmological constant as there was no need for the universe to be static anymore. In fact, even with the presence of a cosmological constant, a matter filled universe is not in a state of stable equilibrium, and hence will not be static. A perturbation of the field equations around the Einstein static universe reveals that any small fluctuation in the metric density will grow unbounded. Heuristically, one can imagine that the cosmological term produces a repelling counter effect to the gravitational attractive force/imploding effect from the matter. In the static universe, these two forces are in balance thus holding space static. If the matter density falls slightly lower, the repulsive force of the cosmological constant causes space to expand which further dilutes the matter, and this effect goes till all matter is diluted away! Similarly, if matter density fluctuation causes it to momentarily grow slightly more than the balance value, its gravitational attraction overcomes the repulsion of the cosmological constant, causing physical space to contract, which in turn increases the matter density further. The process goes on till matter reaches infinite density and space collapses into a singularity. However, cosmological constant has made a comeback, largely due to new data at the turn of the millennium, which suggest that in the current universe, $\ddot{a} > 0$.

Can one try to compute the cosmological constant from first principles? If one identifies the cosmological constant as vacuum energy i.e. the "zero point energy" due to quantum mechanical fluctuation of quantum fields, then the natural value for this in a gravitational theory would be[2]

$$\rho_\Lambda \sim E_P/l_P^3 \approx 10^{124}\, GeVm^{-3}.$$

But current data suggest we live in a critical (flat) universe, with $\rho_c \approx 5.2\ GeVm^{-3}$. This is a huge mismatch, $\frac{\rho_c}{\rho_\Lambda} \sim 10^{-123}$, and is popularly phrased as the cosmological constant problem. Of course our estimation of the vacuum energy was very crude, but this is because we do not have a concrete well defined method in QFT to compute the vacuum energy[3]. In fact in QFT, vacuum energy is not an observable and naive calculations indicate it is formally *infinite* and one usually performs a normal ordering of the Hamiltonian to set the vacuum energy to zero.

[2]The rough logic is that time does not make sense below the Planck time interval, t_P and then maximum fluctuation in energy compatible with energy-time uncertainty principle is, $\frac{\hbar}{t_P}$ which just the Planck energy, E_p. Consider this much energy fluctuation within the smallest possible spatial volume in a theory of gravity, i.e. the Planck volume, l_P^3. Thus an estimate for the zero point fluctuation energy is E_P/l_P^3.

[3]For compact spaces, one can unambiguously define and compute the ground state energy in field theory, dubbed as the Casimir Energy. However for noncompact spaces this is not possible.