PH 6548/EP 4258: Cosmology Lecture 4

Single Component and Multi-component Universes*

November 25, 2019

In the previous lectures we have seen that the dynamics of spatially homogeneous and isotropic universes are described in general relativity by the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho - \frac{\kappa}{a^2 R_0^2}, \tag{1}$$

the matter-energy continuity equation,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0, \tag{2}$$

and an equation of state for each type of matter-energy source,

$$P_w = w\,\rho_w \tag{3}$$

where, P_w and ρ_w are the partial pressure and energy density respectively of that component/type of matter-energy source. The total pressure is then given by the sum over the components,

$$P = \sum_w P_w = \sum_w w\,\rho_w,$$

and total energy density is also given by a sum over all the components/different sources,

$$\rho = \sum_w \rho_w.$$

Evidently, for such a multi-component universe (i.e. one with many different sources of matter-energy), one cannot write down a direct equation relating total pressure, P, and total energy density, ρ. Consequently, solving the fluid continuity equation, (2) becomes very difficult. To simplify the situation we will assume that the different components of matter-energy do not interact, and their energy-momentum is *separately* conserved, i.e. the fluid continuity equation holds *separately for each component*,

$$\dot{\rho}_w + 3\frac{\dot{a}}{a}(\rho_w + P_w) = 0 \Rightarrow \dot{\rho}_w + 3(1+w)\frac{\dot{a}}{a}\dot{\rho}_w = 0.$$

which can be readily solved to obtain the evolution of energy density,

$$\boxed{\rho_w(t) = \frac{\rho_{w,0}}{a(t)^{3(1+w)}}.} \tag{4}$$

*Please communicate any typos spotted to Shubho Roy @ sroy@iith.ac.in.

Here $\rho_{w,0} = \rho_w(t_0)$ is the energy density of matter-energy source component w in the present day universe, i.e. at time, t_0. From this solution, we note that, energy density of matter gets diluted with time as a^{-3}, energy density of radiation falls off with time as a^{-4}, while the energy density of vacuum energy remains constant over time. Curvature term, when treated as another source of matter-energy, and described by an energy density $\rho_\kappa = -\frac{3\kappa}{8\pi G_N R_0^2 a^2}$ evidently falls off with time as a^{-2}. The same conclusion can be arrived at from the solution (4) when one recalls that for curvature, $w = -\frac{1}{3}$. These fall off behaviors are intuitively obvious. For nonrelativistic matter (dust), the number of particles in the universe remains constant as the universe evolves, while the volume goes as a^3 and hence the energy density, which for nonrelativistic matter is same as mass density, must go as

$$\rho_m = \frac{\text{mass}}{\text{volume}} = \frac{\text{constant}}{a^3}.$$

For radiation, say light wave/photons, the energy is given by, $E(t) = h\,\nu(t) = \frac{h}{\lambda(t)}$. As the universe expands, comoving lengths/distances scale as $a(t)$, which implies wavelength of light gets stretched to, $\lambda(t) = a(t)\,\lambda_0$ which further implies the energy of a photon falls off with the scale factor as, $E(t) = \frac{h\nu_0}{a(t)}$. However the volume of the universe grows as $a^3(t)$. Then the energy density of radiation must go as,

$$\rho_r = \frac{\text{energy}}{\text{volume}} = \frac{h\nu_0}{a^4(t)} = \frac{\rho_{r,0}}{a^4(t)}.$$

For vacuum energy i.e. the energy of empty space, the energy density must remain constant because as universe expands it creates energy at a rate proportional to the amount of space created, i.e. a^3, i.e. at the same rate with which volume grows. So the energy density stays same at all times.

To make further progress one has to now solve the Friedmann equation for the scale factor, $a(t)$,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho - \frac{\kappa}{a^2 R_0^2} = \frac{8\pi G_N}{3}\sum_w \frac{\rho_{w,0}}{a^{3(1+w)}}.$$

Again this seems like a very complicated equation to solve because the RHS is a sum of different (inhomogeneous) powers of a., corresponding to different components/sources of matter-energy. So to simplify and build intuition we will consider the situation when one component is overwhelmingly abundant and drives the expansion of the universe, i.e. we will only keep that component in the RHS of the Friedmann equation and set to zero the contribution from the other sources,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\frac{\rho_{w,0}}{a^{3(1+w)}}.$$

Such universes are aptly called *single component universes*. (Note here we have also included the curvature as a source of matter-energy). This assumption might seem unjustified, however as we have seen before, different components of matter-energy get diluted at different rates and it is possible to imagine eras where one component is most abundant while others are not as abundant. For example, at early times in the universe when $a \ll 1$, radiation is the most abundant component because, $\rho_r \sim \frac{1}{a4}$ while at late times, when $a \gg 1$, then vacuum energy is most abundant, as matter, radiation and curvature gets diluted away.

1 Single component universes

At certain times (*epochs*) of cosmic history one component of matter-energy is overwhelmingly abundant (i.e. has energy density much greater than the others) and that component effectively drives the expansion rate of the universe, we can ignore the role of other components, i.e. set their contribution to energy density to zero. The Friedmann equation for such single component universe simplifies greatly,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho_w,$$

where we have also included/regarded as a source of matter energy with energy density and pressure,

$$\rho_\kappa = -\frac{3\kappa}{8\pi G_N R_0^2 a^2}, P_\kappa = \frac{\kappa}{8\pi G_N R_0^2 a^2}.$$

So for curvature component, $w = -\frac{1}{3}$. Since$\rho_w(t)$ has already been solved as a function of the scale factor in Eq. (4), the Friedmann equation for the single component universes turn into,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\frac{\rho_{w,0}}{a(t)^{3(1+w)}},$$

which can be immediately solved,

$$a(t) = \left(\frac{8\pi G_N \rho_{w,0}}{3}\right)^{\frac{1}{3(1+w)}} (t - t_*)^{\frac{2}{3(1+w)}}.$$

$t_*$is the time for which $a = 0$, i.e. space crunches to a point, which is called the ***Big Bang singularity***. By convention, we will set $t_* = 0$. Finally we can trade the constant coefficient for the current cosmic time, t_0 by using the boundary condition, $a(t_0) = 1$. Then the solution to the scale factor in the single component universe is,

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}, w \neq -1. \tag{5}$$

As indicated this solution is not valid when, $w = -1$ i.e. for vacuum energy. The solution to Friedmann equation for vacuum energy is,

$$a(t) = C \exp\left(\sqrt{\frac{8\pi G_N \rho_{w,0}}{3}}\, t\right).$$

Again using the boundary condition, $a(t_0) = 1$, we determine the constant,

$$a(t) = \exp\left[\sqrt{\frac{\Lambda}{3}}\,(t - t_0)\right]. \tag{6}$$

This solution is only valid for a positive cosmological constant, $\Lambda > 0$. Since for this case, $a \to 0$ only when $t \to -\infty$, this universe does not have a beginning, or in other words this is a ***Steady State Universe***.

Source (component)	w	$\frac{\rho_w(t)}{\rho_{w,0}}$	$a(t)$	Type of Universe
Matter (dust)	0	$a^{-3}(t)$	$\propto t^{\frac{2}{3}}$	Big Bang
Radiation	$\frac{1}{3}$	$a^{-4}(t)$	$\propto t^{\frac{1}{2}}$	Big Bang
(Positive) Curvature[1]	$-\frac{1}{3}$	$a^{-2}(t)$	$\propto t$	Steady State (Fake Big Bang)
(Positive) Vacuum Energy[2]	-1	constant	$\propto \exp\left(\sqrt{\frac{\Lambda}{3}}\, t\right)$	Steady State

Table 1: Summary of features of single component universes

The Hubble parameter for these single component universes are,

$$H(t) \equiv \frac{\dot{a}}{a} = \begin{cases} \frac{2}{3(1+w)}\frac{1}{t}, w \neq -1 \\ \sqrt{\frac{\Lambda}{3}}, w = -1. \end{cases}$$

In particular the Hubble constant is related to the current time, t_0 by,

$$H_0 = \begin{cases} \frac{2}{3(1+w)}\frac{1}{t_0}, w \neq -1 \\ \sqrt{\frac{\Lambda}{3}}, w = -1. \end{cases} \tag{7}$$

Thus in terms of the Hubble constant the age of the universe, t_0

$$t_0 = \frac{2}{3(1+w)}\frac{1}{H_0}, w \neq -1. \tag{8}$$

Thus, in these universes if one measures the Hubble parameter today it immediately determines the present age of the universe. Table (1) contains a summary of the expansion history of various different single component dominated universes.

1.1 Galactic Redshifts and Horizon sizes in a single component universe

Consider light emitted from a source (galaxy) at cosmic time, t_e , which is observed by an observer at present cosmic time, i.e. $t_o = t_0$ and is observed at a redshift, z. We have the well known redshift-scale factor relation,

$$1 + z = a^{-1}(t_e)$$

and substituting the scale factor expressions (5) and (6) in the RHS, one has,

$$1 + z = \begin{cases} \left(\frac{t_e}{t_0}\right)^{-\frac{2}{3(1+w)}}, \quad w \neq -1 \\ \exp\left[\sqrt{\frac{\Lambda}{3}}\,(t_0 - t_e)\right], w = -1, \end{cases}$$

[1] This positive curvature empty universe is also known as the ***Milne Universe***. One might suspect, an empty universe has to be Minkowski spacetime, $\mathbb{R}^{3,1}$. Indeed one can show that the Milne universe is Minkowski spacetime in some funny coordinate chart.

[2] This positive vacuum energy universe is also known as the ***Einstein de Sitter Universe***. One might suspect, this represents one half the pure de Sitter spacetime.

which determines the time of emission

$$t_e = \begin{cases} \frac{t_0}{(1+z)^{\frac{3(1+w)}{2}}}, & w \neq -1 \\ t_0 - \sqrt{\frac{3}{\Lambda}} \ln(1+z), & w = -1. \end{cases}$$

In terms of the observed quantities, namely the Hubble constant, H_0 and galactic redshifts, z, the emission time,

$$t_e = \begin{cases} \frac{2}{3(1+w)} H_0^{-1} (1+z)^{-\frac{3(1+w)}{2}}, & w \neq -1 \\ t_0 - H_0^{-1} \ln(1+z), & w = -1. \end{cases} \quad (9)$$

Since the redshift is a monotonic function of time the light was emitted, one can trade the redshifts for time.

The comoving separation between the emitter and observer according to the FLRW metric is,

$$r = \int d\rho = \int_{t_e}^{t_0} \frac{dt}{a(t)}.$$

Using the scale factor expression expressions (5) and (6), we get the respective comoving separation distance,

$$r = \begin{cases} \frac{3(1+w)}{1+3w} t_0 \left[1 - \left(\frac{t_e}{t_0}\right)^{\frac{1+3w}{3(1+w)}}\right], & w \neq -1, -\frac{1}{3} \\ t_0 \ln \frac{t_0}{t_e}, & w = -\frac{1}{3} \\ \sqrt{\frac{3}{\Lambda}} \exp\left[\sqrt{\frac{\Lambda}{3}} (t_0 - t_e)\right], & w = -1. \end{cases}$$

In terms of the observable quantities, the redshifts (using 9) and the Hubble constant (7),

$$r = \begin{cases} \frac{2}{1+3w} H_0^{-1} \left[1 - (1+z)^{-\frac{1+3w}{2}}\right], & w \neq -1, -\frac{1}{3} \\ H_0^{-1} \ln(1+z), & w = -\frac{1}{3} \\ H_0^{-1} (1+z), & w = -1. \end{cases}$$

The physical spatial distance of separation between the source and the observer, $D(t)$ at some cosmic time, t can be extracted from the comoving distance by

$$D(t) = a(t)\, r.$$

Since the present day, scale factor, $a(t_0)$ is unity by definition, the present day distance of separation is then same as the comoving separation,

$$D(t_0) = \begin{cases} \frac{2}{1+3w} H_0^{-1} \left[1 - (1+z)^{-\frac{1+3w}{2}}\right], & w \neq -1, -\frac{1}{3} \\ H_0^{-1} \ln(1+z), & w = -\frac{1}{3} \\ H_0^{-1} (1+z), & w = -1. \end{cases} \quad (10)$$

For low redshift galaxies, $z \ll 1$,

$$D(t_0) = \begin{cases} H_0^{-1} z, & w \neq -1 \\ H_0^{-1}, & w = -1. \end{cases}$$

which leads to Hubble's law of galactic redshifts for the big bang universes ($w \neq -1$),

$$z = H_0 \, D(t_0).$$

However for high redshifts galaxies, $z \gg 1$,

$$D(t_0) = \begin{cases} \frac{2}{1+3w} H_0^{-1}, & w \neq -1 \\ H_0^{-1} \, z, & w = -1, \end{cases}$$

and thus one gets a Hubble-type law for steady state universes now,

$$z = H_0 \, D(t_0).$$

The cosmological horizon distance, d_h is defined to be the farthest distance from which light can reach an observer at current time, t_0. It can be obtained by taking the limit[3], $z \to \infty$, $d_h = \lim_{z\to\infty} D(t_0)$

$$d_h == \lim_{z\to\infty} D(t_0) = \begin{cases} \frac{3(1+w)}{1+3w} t_0, & w \neq -1, -\frac{1}{3} \\ \infty, & w = -1, -\frac{1}{3}. \end{cases}$$

Alternatively, in terms of the Hubble constant, H_0,

$$\boxed{d_h = \begin{cases} \frac{2}{1+3w} \frac{1}{H_0}, & w \neq -1, -\frac{1}{3} \\ \infty, & w = -1, -\frac{1}{3}. \end{cases}}$$

Comments

- For the Einstein de Sitter universe, the horizon size is infinite and one can see galaxies which are infinitely far away. This makes sense because it is a steady state universe. For such a universe there will be no Olbers' paradox, as night sky will be as bright as day sky.

- The situation seems paradoxical for the curvature only (empty) universe. The curvature only universe (Milne universe) seems to have a finite age, t_0 counted from a beginning with a bang at some time in the past chosen by convention to be $t = 0$. And yet, there is no cosmological

[3] Another equivalent way of computing the horizon distance is compute the distance light has traveled since the universe was around. For single component universes beginning with a bang,

$$\begin{aligned} d_h &= a(t_0) \int_0^{t_0} \frac{dt}{a(t)} \\ &= \begin{cases} \frac{3(1+w)}{1+3w} \, t_0, & w \neq -1, -\frac{1}{3} \\ \infty, & w = -\frac{1}{3}. \end{cases} \end{aligned}$$

For the steady state, Einstein de Sitter universe,

$$d_h = a(t_0) \int_{-\infty}^{t_0} \frac{dt}{a(t)} \to \infty.$$

Component of matter-energy	Symbol	Density Parameter, Ω
Baryonic matter	B	$\Omega_{B,0} \approx 0.04$
Dark matter	DM	$\Omega_{DM,0} \approx 0.26$
Total matter (dust)	m	$\Omega_{m,0} \approx 0.3$
Microwave background	CMB	$\Omega_{CMB,0} \approx 5.0 \times 10^{-5}$
Neutrino background	ν	$\Omega_{\nu,0} \approx 3.4 \times 10^{-5}$
Starlight		$\Omega_{starlight} \approx 10^{-6}$
Total radiation	γ	$\Omega_{\gamma,0} \approx 10^{-4}$
Vacuum Energy	Λ	$\Omega_{\Lambda,0} \approx 0.7$
Total		$\Omega_0 \approx 1$

Table 2: Estimates of the density parameters of various components of matter-energy sources in the current universe

horizon! To resolve this paradox, recall that the Milne universe is Minkowski space in funny coordinates. Minkowski space truly speaking has no beginning or end, it is homogeneous in time as well. Hence for the Milne case there is no real/physical big bang, but the big bang is due to a choice of coordinates which are ill-defined at $t = 0$. In reality, light from times earlier than the apparent big bang $t = 0$, can cross thru the $t = 0$ fake singularity and reach the observer.

2 Multi-component Universes

In the previous section, we considered dynamics of an universe populated by a single component (source) of matter-energy. A glance at table (2) containing the best estimates of density parameters reveals our universe is anything but such a single component universe. Our universe contains all kinds of sources, matter in form of baryonic matter and dark matter, radiation in the form of the microwave background, the neutrino background as well as starlight, and finally vacuum energy (perhaps even dark energy). Nevertheless, such a single component model will still be appropriate for describing the dynamics of our universe in epochs or stages where one source of matter-energy is overwhelmingly dominant (abundant) compared to the others. Recall that the energy density of a component or source scales with the scale factor of the universe as,

$$\rho_w \propto \frac{1}{a^{3(1+w)}(t)}$$

Although currently matter ($\Omega_{m,0} \sim 0.3$) and vacuum energy ($\Omega_{\Lambda,0} \sim 0.7$) are more dominant compared to radiation ($\Omega_\gamma \sim 10^{-4}$), the evolution of density suggests that in the early universe, when $a(t) \ll 1$, radiation was *the* most dominant component, followed by matter and finally by vacuum energy. As time went on and the universe expanded, radiation got diluted faster than matter (and vacuum energy). One has,

$$\frac{\rho_m(t)}{\rho_\gamma(t)} = \frac{\rho_{m,0}}{\rho_{\gamma,0}}\, a(t)$$

So at a time $t_{m=\gamma}$, when the scale factor was

$$a_{m=\gamma} = \frac{\rho_{\gamma,0}}{\rho_{m,0}} = \frac{\Omega_{\gamma,0}}{\Omega_{m,0}} \approx 3.0 \times 10^{-4}. \tag{11}$$

radiation and matter energy-densities were equal. This epoch in the history of the universe is dubbed as the era of matter-radiation equality. After this time, matter will become more the more dominant component till at some point it gets diluted to the point that vacuum energy starts dominating. Since the energy-density of vacuum energy remains constant, ultimately in the future, when $a(t) \gg 1$, the dominant component will be vacuum energy. The matter-vacuum energy equality happens at a time, $t_{m=\Lambda}$, when the scale factor is,

$$a\left(t_{m=\Lambda}\right) = \left(\frac{\rho_{m,0}}{\rho_{\Lambda,0}}\right)^{1/3} \approx 0.75.$$

In the intermediate times, we have an universe populated with multiple sources of matter-energy. In today's lecture we analyze such multi-component universes.

The Friedmann equation for multi-component universes is

$$H^2(t) = \frac{8\pi G_N}{3} \sum_w \rho_w(t) - \frac{\kappa}{a^2 R_0^2}.$$

In the present day universe this equation is,

$$H_0^2 = \frac{8\pi G_N}{3} \rho_0 - \frac{\kappa}{R_0^2},$$

where ρ_0 is total energy density of the present day universe. This then gives us the curvature,

$$\Rightarrow \frac{\kappa}{R_0^2} = H_0^2\left(\Omega_0 - 1\right).$$

Substituting form of the curvature back in the Friedmann equation we get,

$$H^2(t) = \frac{8\pi G_N}{3} \sum_w \rho_w(t) - \frac{1}{a^2} H_0^2\left(\Omega_0 - 1\right)$$

or,

$$\frac{H^2(t)}{H_0^2} = \sum_w \Omega_w(t) - \frac{\Omega_0 - 1}{a^2(t)}.$$

Further, substituting $\Omega_w(t) = \Omega_{w,0} a^{-3(1+w)}$ one gets the following form of the Friedmann equation,

$$\begin{aligned} \frac{H^2(t)}{H_0^2} &= \sum_w \frac{\Omega_{w,0}}{[a(t)]^{3(1+w)}} - \frac{\Omega_0 - 1}{a^2(t)} \\ &= \frac{\Omega_{\gamma,0}}{a^4(t)} + \frac{\Omega_{m,0}}{a^3(t)} + \Omega_{\Lambda,0} - \frac{\Omega_0 - 1}{a^2(t)}. \end{aligned}$$

The formal solution to this equation is,

$$H_0\, t = \int_0^a \frac{da}{\left(\frac{\Omega_{\gamma,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 - \Omega_0 + 1\right)^{1/2}}. \tag{12}$$

The integration cannot be performed exactly and the answer cannot be written in a simple analytic form. However one can still perform this integral numerically and plot the solution as a graph of a

vs t.

We have already modeled the epochs of the universe when one component was dominant. For example, the early stages the universe was in a radiation dominated epoch, and after a while by a matter dominated epoch, and after a further while followed by curvature dominated epoch and finally in the far future by a vacuum energy dominated epoch. However in the intermediate stages between two successive such epochs of single component domination, one has a universe where two components are of equal order in energy density.

3 Matter and Curvature

This case is more of a historical curiosity, such a universe was considered by cosmologists in the 1950's when the CMB or vacuum energy was not yet known to exist. For this case, we have to set $\Omega_{\gamma,0} = \Omega_{\Lambda,0} = 0$, and $\Omega_{m,0} = \Omega_0$ in the solution for the scale factor (12),

$$H_0\, t = \int_0^a \frac{da}{\left(\frac{\Omega_0}{a} - \Omega_0 + 1\right)^{1/2}}.$$

This integration can be performed and an analytic answer can be obtained. But one needs to consider three different cases:

- **Positive Curvature case ($\Omega_0 > 1$) :**
  Since $\Omega > 1$, $\kappa = +1$ i.e. a positively curved universe. In this case,

$$H_0 t = \frac{1}{\sqrt{\Omega_0 - 1}} \int_0^a da \sqrt{\frac{a}{\frac{\Omega_0}{\Omega_0 - 1} - a}}$$

  Making the substitution,

$$a = \frac{\Omega_0}{\Omega_0 - 1} \sin^2 x, \tag{13}$$

  we obtain the solution,

$$H_0 t = \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \left(x - \frac{1}{2} \sin 2x \right) \tag{14}$$

  We will leave the solution in this parametric form where $a = a(x)$ and $t = t(x)$, instead of eliminating x to express a as a function of t, $a = a(t)$. It is evident that t is a monotonic function of x and hence one can regard x itself as a proxy for time. When $x = 0$ i.e. $t = 0$, one has $a = 0$ i.e. the Big Bang singularity when the universe comes into existence. But $a \to 0$ again as $x = \pi$. Thus the universe collapses to a singularity at a time in the future - the so called ***Big Crunch*** singularity. The cosmic time, t_{crunch}, when the universe collapses into a big crunch singularity is,

$$t_{crunch} = \frac{\pi\, \Omega_0}{(\Omega_0 - 1)^{3/2}} H_0^{-1}. \tag{15}$$

  After the bang this universe expands for a while, hits a maximum and then contracts and collapses to a crunch. The maximum value of the scale factor is reached when $x = \pi/2$,

$$a_{max} = \frac{\Omega_0}{\Omega_0 - 1}. \tag{16}$$

Since $\sin x$ is symmetric about the point, $x = \pi/2$. The curve of the scale factor as a function of cosmic time (or x) is symmetric about the maxima value. So the contraction phase is a time-reversed image of the expansion phase.

- **Negative Curvature case** ($\Omega_0 < 1$) **:**
In this case, $\kappa = -1$, i.e. a negatively curved universe. For this case,

$$H_0\, t = \int_0^a \frac{da}{\left(\frac{\Omega_0}{a} + 1 - \Omega_0\right)^{1/2}}.$$

In parametric form,

$$a = \frac{\Omega_0}{1-\Omega_0} \sinh^2 x, \tag{17}$$

$$H_0 t = \frac{\Omega_0}{(1-\Omega_0)^{3/2}} \left(\frac{\sinh x}{2} - x \right). \tag{18}$$

Just as in the previous (overdense) case, t is a monotonic function of x and we may use x as a proxy for time. The scale factor vanishes as $x = 0$ (same as $t = 0$), which is the ***Big Bang*** singularity. However in this case there is no crunch, and the universe continues to expand forever, $a \to \infty$. Likening the universe to be an adiabatically expanding gas of matter, at infinite volume (scale factor) the temperature hits absolute zero. Such a fate of the universe is called the ***Big Chill*** or the ***Big Freeze***.

- **Flat case** ($\Omega_0 = 1$) **:**
In this case, $\kappa = 0$ i.e. spatially flat universe. But this case does not merit a separate discussion since this a single component universe (nothing but matter). We already know the fate of this universe, and the scale factor is given by,

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}.$$

Just like the underdense case, this universe begins with a ***Big Bang*** at $t = 0$ and ends in a ***Big Chill/Freeze***.

4 Matter and Vacuum Energy

The current epochs of the universe is dominated by matter and vacuum energy, as evident from table (2). For this case, we have to set $\Omega_{\gamma,0} = 0$, $\Omega_0 = 1$ (no curvature) and $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$ in the solution for the scale factor (12),

$$H_0\, t = \int_0^a \frac{da}{\left[\frac{\Omega_{m,0}}{a} + (1 - \Omega_{m,0})\, a^2\right]^{1/2}}$$

To make further analysis we will have to consider two different cases.

- **Positive Cosmological Constant** ($\Omega_{m,0} < 1$)

  In this case, we have a positive cosmological constant, $\Lambda > 0$. The result of the integration for this case is,

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left(\sqrt{1+\frac{1-\Omega_{m,0}}{\Omega_{m,0}}a^3} + \sqrt{\frac{1-\Omega_{m,0}}{\Omega_{m,0}}}\, a^{3/2}\right),$$

  inverting which we obtain,

$$a(t) = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}\sqrt{1-\Omega_{m,0}}\, H_0\, t\right). \tag{19}$$

  Thus the scale factor is a monotonically increasing function of the cosmic time. Since, $a = 0$ for $t = 0$, this universe begins in a Big Bang. For early times, $t \ll H_0^{-1}$,

$$a(t) \approx (\frac{3}{2}\Omega_{m,0}^{1/2} H_0\, t)^{2/3}$$

  which is indeed a matter dominated universe. As $H_0\, t \gg 1$ the universe gets vacuum energy dominated, the scale factor begins to grow exponentially,

$$a(t) \approx \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3} \exp\left(\sqrt{1-\Omega_{m,0}}\, H_0\, t\right).$$

  Thus the universe again ends in a Big Chill.

Exercise: Age of the Universe
Compute the age of the universe, using the observed values of using the observed values of $\Omega_{m,0}$ and H_0.

Excercise: Matter-Vacuum energy equality
Compute the time, $t_{m=\Lambda}$ when the energy density in matter and vacuum energy were equal, ρ_m/ρ_Λ using the observed values of $\Omega_{m,0}$ and H_0.

- **Negative cosmological constant** ($\Omega_{m,0} > 1$):

  In this case, $\Omega_{\Lambda,0} = 1 - \Omega_{m,0} < 0$ i.e. a negative cosmological constant. In this case we have the solution,

$$H_0\, t = \int_0^a \frac{da}{\left[\frac{\Omega_{m,0}}{a} - (\Omega_{m,0} - 1)\, a^2\right]^{1/2}}$$

  This integral can be solved exactly,

$$H_0 t = \frac{2}{3\sqrt{\Omega_{m,0} - 1}} \sin^{-1}\left(\sqrt{\frac{\Omega_{m,0} - 1}{\Omega_{m,0}}}\, a^{3/2}\right),$$

or inverting,

$$a(t) = \left(\frac{\Omega_{m,0}}{\Omega_{m,0}-1}\right)^{1/3} \sin^{2/3}\left(\frac{3}{2}H_0\sqrt{\Omega_{m,0}-1}\,t\right). \tag{20}$$

In addition to the big bang singularity at $t = 0$, we note that at a later time,

$$t_{crunch} = \frac{2\pi}{3H_0\sqrt{\Omega_{m,0}-1}}, \tag{21}$$

$a \to 0$, i.e. the universe hits a big crunch singularity at time, t_{crunch}. Thus, a negative cosmological constant acts as an attractive/ implosive force which causes the universe to contract and ultimately collapse to a crunch singularity. The maximum scale factor of this universe is,

$$a_{max} = \left(\frac{\Omega_{m,0}}{\Omega_{m,0}-1}\right)^{1/3}.$$

Exercise: Which universe is more short lived i.e. which one crunches faster - (a) the matter and positive curvature universe ($\Omega_0 > 1$), or (b) the matter and negative vacuum energy universe ($\Omega_{m,0} > 1$). For each case assume a nominal value for matter (dust) density parameter $\Omega_0 = \Omega_{m,0} = 1.1$.

5 Matter and Radiation Universe

Finally let's consider the early stages of the expansion history of the universe when matter and radiation were the ones with high energy density. Setting the curvature and vacuum energy in such an universe to zero, the cosmic time as a function of the scale factor is given by,

$$\begin{aligned} H_0\,t &= \int_0^a \frac{da}{\left(\frac{\Omega_{\gamma,0}}{a^2}+\frac{\Omega_{m,0}}{a}\right)^{1/2}} \\ &= \frac{1}{\sqrt{\Omega_{r,0}}}\int_0^a \frac{a\,da}{\sqrt{1+\frac{a}{a_{m=\gamma}}}}, \quad a_{m=\gamma} = \frac{\Omega_{\gamma,0}}{\Omega_{m,0}} \\ &= \frac{4a_{m=\gamma}^2}{3\sqrt{\Omega_{r,0}}}\left[\left(\frac{a}{2a_{m=\gamma}}-1\right)\sqrt{1+\frac{a}{a_{m=\gamma}}}+1\right]. \end{aligned} \tag{22}$$

For early times, i.e. well before radiation-matter equality $\frac{a}{a_{m=\gamma}} \ll 1$,

$$a \approx \left(2\sqrt{\Omega_{\gamma,0}}\,H_0\,t\right)^{1/2},$$

which is characteristic of a radiation-dominated epoch, as one would expect. For late times, $H_0 t \gg 1$

$$a(t) \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}}\,H_0\,t\right)^{2/3},$$

which indeed as expected is the characteristic matter-dominated time dependence.

Exercise: Compute the cosmic time of matter-radiation equality, $t_{m=\gamma}$, using best estimates of $\Omega_{m,0}, \Omega_{\gamma,0}$, and H_0.