REPRESENTATIONS OF THE AUTOMORPHISM GROUP OF A GRAPH

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Abstract. Given a graph G, a natural group structure arising from G is its set of automorphisms on the vertices together with the operation of function composition. A natural vector space arising from G is its Edge Space, $\mathcal{E}(G)$, defined as the formal vector space generated by the edges of G with coefficients in $\mathbb{C}$. Two natural subsapces of $\mathcal{E}(G)$ that arise from G are its cycle space and cut space — the null space and row space of the incidence matrix of G. We investigate representations of $\mathrm{Aut}(G)$ over $\mathcal{E}(G)$, $\mathrm{Cyc}(G)$ and $\mathrm{Cut}(G)$. We show that for the graphs K_3, K_4, and K_5, the representation $\rho : \mathrm{Aut}(G) \to \mathcal{E}(G)$ by $\mathrm{Aut}(G)$ acting on the vertices has the irreducible decompostion $\mathrm{Cyc}(G) \oplus \mathrm{Cut}(G)$. With the support of computational evidence, we conjecture that this irreducible decomposition holds for every complete graph on n vertices.

1. Introduction

Representation theory is a rich subject with many applications to areas in math as well as areas outside of math, including physics, coding theory, and more recently computer science. As representation theory has seen much success in the field of algebra, a natural topic to explore is how representation theory can be used in the context of algebraic graph theory. Some exciting work has already been done in this area. For example, given a group H and a graph G, deciding whether H has a nontrivial representation $\rho\colon H \to \mathrm{Aut}(G)$ is known to being closely related to the graph isomophism problem [1]. In addition, a well known result, *Frucht's Theorem*, states that for every finite group H, there exists infinitely many non-isomorphic simple connected graphs G such that $H \cong \mathrm{Aut}(G)$. The relation between a group and the automorphism group of its Cayley graph has been well studied in the literature. Results on the decomposition of graph eigenspaces into reducible and irreducible subspaces has been presented in Biggs's *Algebraic Graph Theory* [2]. Other work on Singular Graphs (graphs whose adjacency matrix is singular) has been done by Alu Al-Tarimshawy in his doctorate thesis [3]. In addition, work by Gregory Berkolaiko and Wen Liu showed that the eigenspaces of symmetric graphs are not typically irreducible [4].

Despite these successes, there does not appear to be results studying the representation of the automorphism group of a graph G acting on its natural vectors spaces $\mathrm{Cyc}(G)$ and $\mathrm{Cut}(G)$. This will be the main focus of this paper.

2. Automorphisms of Graphs

We consider graphs which are finite and loopless. Although multiedges are allowed, for the purposes of this paper they do not change any results, and so we do not consider multigraphs. Let G be a graph with n vertices. A natural action on G is to permute these vertices, which in turn permutes the corresponding edges. Any permutation which preserves edge structure is called an *automorphism* of G.

Definition 2.1. An *automorphism* of a graph G is a permutation on the vertices

$$\sigma\colon V(G) \to V(G)$$

such that

$$\forall uv \in E(G),\ \sigma(u)\sigma(v) \in E(G).$$

It will be convienent to label the vertices as elements of $\{1, \dots, n\}$, and view σ as an element of S_n, the symmetric group on n objects. For clairity, we now provide an example of an automorphism on a graph.

Example 2.2. For the graph in Figure 1 below, we see that (12) is an automoprhism, yet (14) is not, as $\{1, 2\} \notin E(G)$.

The set of all automorphisms of G, denoted $\mathrm{Aut}(G)$, forms a group under function composition. It can be checked that the following graphs have the corresponding automorphism groups:

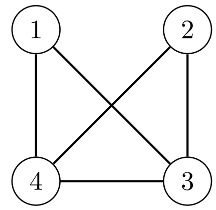

FIGURE 1. (12) is an automorphism, while (14) is not.

- $\mathrm{Aut}(K_n) \cong S_n$
- $\mathrm{Aut}(C_n) \cong D_{2n}$, the Dihedral Group of a regular n-gon, $n \geq 3$
- $\mathrm{Aut}(P_n) \cong \mathbb{Z}/2\mathbb{Z}$, corresponding to reversing the direction of the path
- $\mathrm{Aut}(\mathrm{Star}_n) \cong S_{n-1}$
- $\mathrm{Aut}(\mathrm{PetersonGraph}) \cong S_5$

It should be noted that two nonisomorphic graphs may correspond to the same automorphism group. For example, the star graph on n vertices and the complete graph on $n-1$ vertices both have automorphism group S_{n-1}.

We now introduce the edge space of a graph G as the vector space over $\mathbb{C}$ formally generated by the edges of G. To do so, we consider an orientation on the edges of G and introduce a standard ordering on the edges as follows: for each edge of G, assign the edge an orientation from the smallest vertex to the largest vertex. We now order the edges as follows: for edges (u, v) and (a, b), if $u < a$, or if $u = a$ and $v < b$, then $(u, v) < (a, b)$. We call this the *standard orientation ordering* of the edges. An example is provided for K_4 in Figure 2.

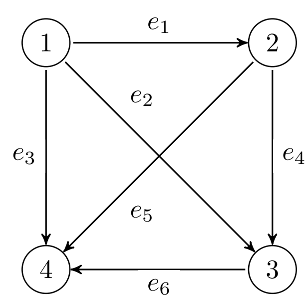

FIGURE 2. K_4 with the standard orientation ordering of edges.

With the standard ordering of the edges we now construct the Incidence Matrix of a graph, which we define below.

Definition 2.3. Let $V(G) = \{ v_1, \ldots, v_n \}$ and $E(G) = \{ e_1, \ldots, e_m \}$. Then the *Incidence Matrix* of a graph G, denoted $A(G)$ or just A when G is understood, is the $n \times m$ matrix defined as follows:

$$A_{ij} = \begin{cases} 1, & v_i \text{ is the head of } e_j \\ -1, & v_i \text{ is the tail of } e_j \\ 0, & \text{otherwise} \end{cases}$$

Example 2.4. For example, the incidence matrix of K_4 with the standard orientation ordering of the edges is given by:

$$A = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

We can now define the edge space of G. where the set of edges of G form the basis elements, and the coefficients are over $\mathbb{C}$.

Definition 2.5. Let G be a graph with m edges $\{ e_1, e_2, \dots, e_m \}$. Let the edges have the standard orientation ordering. Then the *edge space* of G, denoted $\mathcal{E}(G)$, is the set of all formal linear combinations of edges of G with coefficients in $\mathbb{C}$.

We can therefore think of a set of edges in G as a vector v with m entries in $\mathcal{E}(G)$, where the i-th entry in v corresponds to the weight of edge e_i in G. If the i-th entry of v is negative, this means e_i appears in G with opposite orientation (i.e., from a higher labeled vertex to a lower labeled vertex). Therefore, a vector $v \in \mathcal{E}(G)$ has the form:

$$v = c_1(i_1, j_1) + c_2(i_2, j_2) + \cdots + c_m(i_m, j_m),$$

where $c_k \in \mathbb{C}$ and $i_k, j_k \in V(G)$ for $1 \leq k \leq m$.

Two important vector spaces arise from the graph G as subspaces of $\mathcal{E}(G)$: the cycle space and the cut space. Inutitively, each vector in the cycle space corresponds to a set of edges formed by a linear combination of cycles in G, whereas each vector in the cut space correpsonds to a set of edges formed by a linear combination of cuts in G. These spaces correspond to the Null Space and Row Space of the incidence matrix of G. In fact, these are usually taken as the definitions for $\mathrm{Cyc}(G)$ and $\mathrm{Cut}(G)$.

Definition 2.6. The *cycle space* of a graph G, denoted $\mathrm{Cyc}(G)$ is the null space of the incidence matrix of G. That is,

$$\mathrm{Cyc}(G) = \mathrm{Null}(A(G)).$$

The *cut space* of a graph G is the row space of the incidence matrix of G. That is,

$$\mathrm{Cut}(G) = \mathrm{Col}(A^T(G)).$$

Since $\mathrm{Cyc}(G)$ and $\mathrm{Cut}(G)$ are a nullspace and rowspace respectively, they are subspaces of $\mathcal{E}(G)$.

For an intuitive understanding of why these definitions make sense, if $v_k \in V(G)$ is a vertex in a cycle C, then there exist edges e_i and e_j of C such that v_k is the head of e_i and the tail of e_j. Thus the k,i-th entry of $A(G)$ is 1, and the k,j-th entry of $A(G)$ is -1. Since v_k is incident to no other edges of C, we have that the dot product of the k-th row of $A(G)$ and the vector representing C is equal to

$$\sum_{j\colon v_k \in e_j} A_{k,j} = (1) + (-1) = 0$$

Therefore $C \in \mathrm{Null}(A)$. As the elements of $\mathrm{Cyc}(G)$ are linear combinations of cycles, this shows any element of $\mathrm{Cyc}(G)$ is in $\mathrm{Null}(A)$. For the inclusion in the other direction, and for a similar argument for $\mathrm{Cut}(G)$, see Chapter 3 of [2].

3. Representations of the automorphism group of a graph

We can look at the representations of the automorphism group of a graph over the vector spaces $\mathcal{E}(G)$, $\mathrm{Cyc}(G)$, and $\mathrm{Cut}(G)$. For now, we will restrict our attention to $\mathcal{E}(G)$. Recall for a group H and vector space V, a representation is a homomorphism $\rho\colon H \to \mathrm{Aut}(V)$. In our case, we let $H = \mathrm{Aut}(G)$ and $V = \mathcal{E}(G)$. By letting $\mathrm{Aut}(G)$ act on the vertices by permutation, the vectors in $\mathcal{E}(G)$ are linearly transformed. Therefore each $\sigma \in \mathrm{Aut}(G)$ corresponds to a matrix ρ_σ which acts on $\mathcal{E}(G)$. Thus $\sigma \mapsto \rho_\sigma$ is a group representation. We prove this formally below. To do so, we introduce some notation.

Recall a vector $v \in \mathcal{E}(G)$ has the form:

$$v = c_1(i_1, j_1) + c_2(i_2, j_2) + \cdots + c_m(i_m, j_m),$$

where $c_k \in \mathbb{C}$ and $i_k, j_k \in V(G)$ for $1 \leq k \leq m$. Then we have that $\sigma \in \mathrm{Aut}(G)$ acts on a vector $v \in \mathcal{E}(G)$ as follows:

$$\sigma(v) = c_1\left(\sigma(i_1), \sigma(j_1)\right) + c_2\left(\sigma(i_2), \sigma(j_2)\right) + \cdots + c_m\left(\sigma(i_m), \sigma(j_m)\right).$$

It is quite clear to see for $u, v \in \mathcal{E}(G)$, we have

$$\sigma(u + v) = \sigma(u) + \sigma(v).$$

Therefore we have that σ acts linearly on $\mathcal{E}(G)$. Therefore each $\sigma \in \mathrm{Aut}(G)$ can be mapped to a matrix ρ_σ corresponding to this linear action. Note since $\mathrm{Cyc}(C), \mathrm{Cut}(G) \subseteq \mathcal{E}(G)$, we have σ acts linearly on $\mathrm{Cyc}(C)$ and $\mathrm{Cut}(G)$ as well. We now prove that $\rho\colon \mathrm{Aut}(G) \to \mathcal{E}(G)$ defined by $\sigma \mapsto \rho_\sigma$ is indeed a representation.

Theorem 3.1. Let G a graph and $\rho\colon \mathrm{Aut}(G) \to \mathcal{E}(G)$ by $\sigma \mapsto \rho_\sigma$ as described above. Then ρ is a representation of $\mathrm{Aut}(G)$.

Proof. Let G have m edges $\{e_1, e_2, \ldots, e_m\}$ in the standard orientation ordering. Let $\sigma \in \mathrm{Aut}(G)$, and let $\sigma(v)$ denote the vector v after σ has acted on $V(G)$. Define a matrix ρ_σ as follows: for $i = 1, \ldots, m$, let the i-th column of ρ_σ equal $\sigma(e_i)$. Then we have established the mapping $\sigma \mapsto \rho_\sigma$. We now show ρ is a homomorphism.

Let $\sigma, \tau \in \mathrm{Aut}(G)$. We must show $\rho_{\sigma\tau} = \rho_\sigma\rho_\tau$. By definition of the mapping $\sigma \to \rho_\sigma$, we have:

$$\rho_{\sigma\tau} = \begin{bmatrix} \sigma\tau(e_1) & \sigma\tau(e_2) & \ldots & \sigma\tau(e_m) \end{bmatrix}.$$

Thus the i-th column of $\rho_{\sigma\tau}$ is equal to:

$$\begin{aligned}
(\rho_{\sigma\tau})_i &= \sigma\tau(e_i) \\
&= \sigma\left(\begin{bmatrix} \tau(e_i)_1 \\ \tau(e_i)_2 \\ \vdots \\ \tau(e_i)_m \end{bmatrix}\right), \text{ where } \tau(e_i)_j \text{ is the } j\text{-th entry of } \tau(e_i) \\
&= \sigma\left(\tau(e_i)_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \tau(e_i)_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \tau(e_i)_k \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right) \\
&= \sigma\left(\tau(e_i)_1 e_1 + \tau(e_i)_2 e_2 + \cdots + \tau(e_i)_m e_m\right) \\
&= \tau(e_i)_1\sigma(e_1) + \tau(e_i)_2\sigma(e_2) + \cdots + \tau(e_i)_m\sigma(e_m), \text{ by linearity} \\
&= \sigma(e_1)\tau(e_i)_1 + \sigma(e_2)\tau(e_i)_2 + \cdots + \sigma(e_m)\tau(e_i)_m.
\end{aligned}$$

Now we have that

$$\rho_\sigma\rho_\tau = \begin{bmatrix} \sigma(e_1) & \ldots & \sigma(e_m) \end{bmatrix} \begin{bmatrix} \tau(e_1) \ldots \tau(e_m) \end{bmatrix},$$

and the i-th column of $\rho_\sigma\rho_\tau$ is given by the product of ρ_σ and the i-th column of ρ_τ. That is:

$$\begin{aligned}
(\rho_\sigma\rho_\tau)_i &= \rho_\sigma(\rho_\tau)_i \\
&= \begin{bmatrix} \sigma(e_1) & \ldots & \sigma(e_m) \end{bmatrix} \begin{bmatrix} \tau(e_i)_1 \\ \tau(e_i)_2 \\ \vdots \\ \tau(e_i)_m \end{bmatrix} \\
&= \sigma(e_1)\tau(e_i)_1 + \sigma(e_2)\tau(e_i)_2 + \cdots + \sigma(e_m)\tau(e_i)_m.
\end{aligned}$$

Thus the i-th column in each matrix is equal, so $\rho_{\sigma\tau} = \rho_\sigma\rho_\tau$. □

Corollary 3.2. Let $\mathcal{E}(G)$ be the representation as described above. Then $\mathrm{Cyc}(G)$ and $\mathrm{Cut}(G)$ are subrepresentations.

Proof. Since $\mathrm{Cyc}(G)$ and $\mathrm{Cut}(G)$ are subspaces of $\mathcal{E}(G)$, we just need to show that each is invariant under $\sigma \in \mathrm{Aut}(G)$. Since σ is an automorphism of G and corresponds to a relabeling of vertices, the edge structure of any edge set of G is unchanged. Thus a a cycle in G remains a cycle; a cut remains a cut. Therefore $\mathrm{Cyc}(G)$ and $\mathrm{Cut}(G)$ are invariant under σ. □

We now give an example of the representation of K_4 over its cycle space. Recall the incidence matrix of K_4 is given by A in example 2.4. We find the null space of A has basis given by

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

which corresponds to the cycles

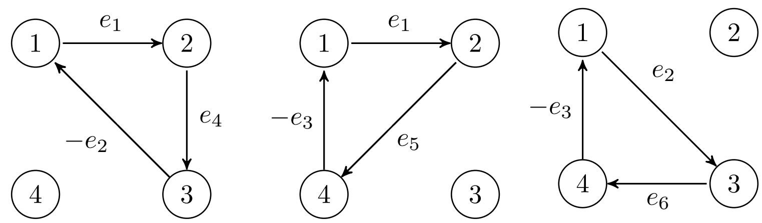

FIGURE 3. Cycle basis for K_4

Let the vector representation in $\mathcal{E}(K_4)$ of these cycles be represented by c_1, c_2, c_3, respectively. Since these are basis elements of $\mathrm{Cyc}(K_4)$, when viewed as vectors in $\mathrm{Cyc}(K_4)$ they correspond to the vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, respectively. After permutation by $\sigma = (1\ 2)$, these cycles become:

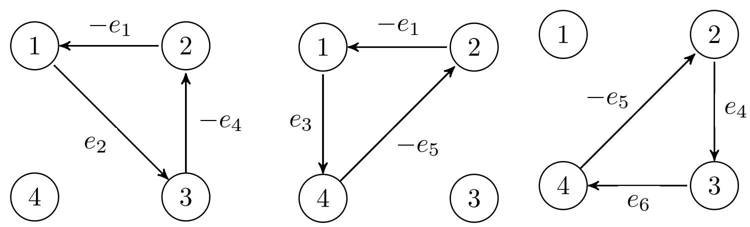

FIGURE 4. (1 2) acting on the cycle basis for K_4

which, when viewed as elements of $\mathcal{E}(K_4)$, are the vectors $(-1,1,0,-1,0,0)$, $(-1,0,1,0,-1,0)$, $(0,0,0,1,-1,1)$ respectively. These vectors can be written as:

$$\begin{bmatrix}-1\\1\\0\\-1\\0\\0\end{bmatrix} = -\begin{bmatrix}1\\-1\\0\\1\\0\\0\end{bmatrix} = -c_1$$

$$\begin{bmatrix}-1\\0\\1\\0\\-1\\0\end{bmatrix} = -\begin{bmatrix}1\\0\\-1\\0\\-1\\0\end{bmatrix} = -c_2$$

$$\begin{bmatrix}0\\0\\0\\1\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\0\\1\\0\\0\end{bmatrix} - \begin{bmatrix}1\\0\\-1\\0\\-1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\-1\\0\\0\\1\end{bmatrix} = c_1 - c_2 + c_3.$$

Therefore, when written as vectors in the Cycle Space with basis $\{ c_1, c_2, c_3 \}$, we have the permuted cycles correspond to the vectors $(-1,0,0)$, $(0,-1,0)$, and $(1,-1,1)$, respectively. Therefore, we have $\sigma = (1\ 2)$ acts as the matrix

$$\begin{bmatrix}-1 & 0 & 1\\0 & -1 & -1\\0 & 0 & 1\end{bmatrix}$$

on $\mathrm{Cyc}(K_4)$.

4. Irreducibility of $\mathrm{Cyc}(K_n)$ and $\mathrm{Cut}(K_n)$

Since $\mathrm{Cyc}(K_n)$ and $\mathrm{Cut}(K_n)$ are orthogonal, we have that $\mathcal{E}(K_n) = \mathrm{Cyc}(K_n) \oplus \mathrm{Cut}(K_n)$. We show that for $n = 3, 4, 5$ that this is indeed the irreducible decomposition of $\mathcal{E}(K_n)$. We do this by computing the inner product of the character function with itself. We start with a well known lemma in representation theory (see Chapter 1 of [5]).

Lemma 4.1. Let ρ be a representation with its irreducible decomposition given by

$$\rho = \rho_1^{\oplus m_1} \oplus \rho_2^{\oplus m_2} \oplus \ldots \oplus \rho_l^{\oplus m_l}.$$

Let χ be the character function of ρ. Then

$$(\chi, \chi) = \sum_{i=1}^{l} m_l^2.$$

From this we see that if $(\chi, \chi) = 2$, then each m_i must equal 0 or 1, and for the sum to equal 2, we must have ρ is the sum of two irreducible representations. Using this observation, we can now prove the following result.

Theorem 4.2. $\mathrm{Cyc}(K_n)$ and $\mathrm{Cut}(K_n)$ are irreducible for $n = 3, 4, 5$.

Proof. Let χ be the character function of $\rho\colon \mathrm{Aut}(K_n) \to \mathcal{E}(K_n)$. Since ρ_σ is a permutation matrix, its diagonal elements are 1 (which corresponds to a fixed edge), -1 (which corresponds to a reversed edge), and 0. Then

$$\chi(\sigma) = \mathrm{Tr}(\rho_\sigma) = \#\text{ fixed edges} - \#\text{ reversed edges}.$$

We start with $n = 3$. In this case, $\mathrm{Aut}(K_3) \cong S_3$. We have the following chart

cycle types	e	(1 2)	(1 2 3)
number of cycles of this type	1	3	2
number of edges fixed	3	0	0
number of edges reversed	0	1	0
χ	3	-1	0

Now because σ and σ^{-1} have the same cycle type, we have $\chi(\sigma) = \chi(\sigma^{-1})$. Recall for a representation of a group Γ the inner product (χ, χ) is given by:

$$(\chi, \chi) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi(g)\chi(g^{-1})$$

We therefore have:

$$\begin{aligned}(\chi, \chi) &= \frac{1}{|S_3|} \sum_{\sigma \in S_3} \chi(\sigma)\chi(\sigma^{-1}) \\ &= \frac{1}{6} \sum_{\sigma \in S_3} \chi(\sigma)^2 \\ &= \frac{1}{6}\left(1(3)^2 + 3(-1)^2 + 2(0)^2\right) \\ &= 2\end{aligned}$$

Therefore, by lemma 4.1, we have ρ is the direct sum of two irreps. If $\mathrm{Cyc}(K_3)$ or $\mathrm{Cut}(K_3)$ were not irreps, then ρ would be the direct sum of more than two irreps, a contradiction. Therefore $\mathrm{Cyc}(K_3)$ and $\mathrm{Cut}(K_3)$ are irreducible.

A similar proof works for K_4 and K_5. $\square$

5. Conjectures and further work

The method above of computing the inner product of $\mathcal{E}(K_n)$ with itself may be able to be generalized to an arbitrary value of n to prove that $\mathrm{Cyc}(K_n)$ and $\mathrm{Cut}(K_n)$ are irreducible under this representation. Indeed, a calculation of the characters for the representation over $\mathrm{Cyc}(K_n)$ up to $n = 10$ have shown that these representations are in the character table of S_n, suggesting $\mathrm{Cyc}(K_n)$ is irreducible. We therefore make the following conjecture:

Conjecture 5.1. The representation $\rho\colon \mathrm{Aut}(K_n) \to \mathcal{E}(K_n)$ given by the action of permuting the vertices has the irreducible decomposition

$$\mathcal{E}(K_n) = \mathrm{Cyc}(K_n) \oplus \mathrm{Cut}(K_n).$$

A natural question that follows is which irreps of S_n are $\mathrm{Cyc}(K_n)$ and $\mathrm{Cut}(K_n)$. Computations have shows that for $n = 3, 4, 5, 6$ that $\mathrm{Cyc}(K_n)$ is the second exterior power of the standard representation. We therefore make the following conjecture:

Conjecture 5.2. $\mathrm{Cyc}(K_n) \cong \wedge^2(\text{standard})$

Plans for further work other than trying to prove the above conjectures include identifying which irrep of S_n that $\mathrm{Cut}(K_n)$ is isomoprhic to, as well as investigating the irreducible decomposition of $\mathcal{E}(G)$ for graphs other than K_n.

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