# AN INTRODUCTION TO REPRESENTATION STABILITY

ROBERTA SHAPIRO

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Abstract. In 2013, Thomas Church and Benson Farb published an extensive survey on homological and representation stability aimed at mathematicians specializing in both geometry and algebra. Our goal today is to distill the idea of representation stability, working toward understanding modern results. We will begin by introducing the concept of stability in an intuitive manner, continue by describing the braid group, and finally introduce a result of Church and Farb's on the representation stability of the cohomology of the pure braid group. After the presentation, if any new concepts pique one's interest or are unclear, there are more sections with further discussion on several subject matters.

## 1. Guide for the Presentation

In this section, we provide an outline for the presentation, as well as some definitions (for your reference during the presentation) and further readings in case you are interested in learning more. Key words and phrases are colored to help find any supplementary information quickly during the lecture.

### 1.1. Stability and Representations.

**Definition 1** (Representation Stability). *Let $(G_n, V_n, \phi_n)$ be a sequence of a triples of groups $G_n$ with $G_n \leq G_{n+1}$ for all $n$, representations $V_n$, and linear functions $\phi_n : V_n \to V_{n+1}$ such that for all $g \in G_n$, the following diagram commutes:*

$$\begin{tikzcd} V_n \arrow[r, "\phi_n"] \arrow[d, "\rho_{n,g}"] & V_{n+1} \arrow[d, "\rho_{n+1,g}"] \\ V_n \arrow[r, "\phi_n"] & V_{n+1} \end{tikzcd}$$

*Note that since $g \in G_n$, we can consider the image of $g$ under inclusion into $G_{n+1}$ to make the right down arrow make sense.*

***Henceforth, we will assume that** $G_n = S_n$.*

*Then, this sequence of representations is **stable** if:*

1. *each $\phi_n$ is injective,*
2. *the $G_{n+1}-$orbit [1] of $\phi_n(V_n)$. spans $V_{n+1}$, and*
3. *if the decomposition into irreducible representations (indexed by padded partitions, defined on the next page) is:*

$$V_n = \bigoplus_{\lambda} c_{n,\lambda} V(\lambda),$$

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[1] That is, the image of $\rho_{n+1,g}$ for all $g \in G_{n+1}$

*then for any fixed $\lambda$, $c_{n,\lambda}$ is constant for large enough $n$.*

*This sequence is* ***uniformly stable*** *if (1) and (2) are satisfied, and*

(3') *if the decomposition into irreducible representations (indexed by padded partitions) is:*

$$V_n = \bigoplus_\lambda c_{n,\lambda} V(\lambda),$$

*then there exists $N$ such that for all $m, n \geq N$, $c_{n,\lambda} = c_{m,\lambda}$ for all $\lambda$.*

In the presentation itself, we will be more interested in uniform stability and ignore regular stability.

The definition of representation stability uses the idea of a padded partition to generalize a partition $\lambda$ to apply for any $n$. Now we will intuitively describe a **padded partition**.

We can describe a partition of an integer $n$ using a Young tableau; call its shape $\lambda$. Then, if we are given $n$, we can recover the entirety of $\lambda$ if we are given all but the first row. Thus, if we vary $n$, $\lambda$ can be seen as a partition of *any* $n$ (large enough) by adjusting the length of the top row. We denote the irreducible representation associated with padded partition $\lambda$ by $V(\lambda)$, noting that $V(\lambda)$ can be an irreducible representation for *almost any* $n$.

**Example 2.** *Consider $n = 6$. Then, one partition of $n$ is $\lambda = (3, 2, 1)$. To generalize $\lambda$ to be a partition of some arbitrary $n$, we remove the first entry (or row in a Young tableau) to get $(-, 2, 1)$. An example of this padded partition can be found in Figure 1.*

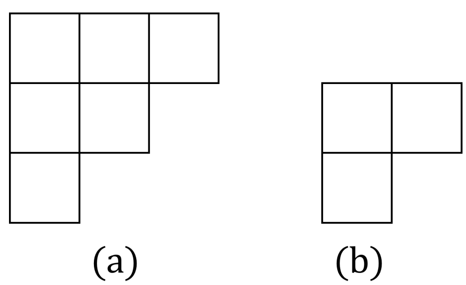


FIGURE 1. In (a), we see the entirety of $\lambda$, a partition of $n = 6$. In (b), we see a padded partition $\lambda$, which we can generalize to be a partition of any $n \geq 5$ by adjusting the number of boxes in the top row.

Also, we note that the concepts of "stable" and "uniformly stable" mimic the ideas of uniform continuity versus continuity.

1.2. **An Example.** Consider the sequence of triples $(S_n, \mathbb{R}^n, i_n : \mathbb{R}^n \to \mathbb{R}^{n+1})$, where $i_n$ is inclusion. We will drop the $n$ when it is clear what $n$ is. We claim that this sequence of $S_n-$representations that arises from the permutation action is uniformly stable.

First, $X_n \leq S_{n+1}$ for all $n$. Second, the diagram in the definition of representation stability commutes (which we can see since the image of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ looks the same as $\mathbb{R}^n$ in a very clear way).

Furthermore, each inclusion $i$ is injective (fulfilling requirement (1)) and the $S_{n+1}-$orbit of $i(\mathbb{R}^n$ spans $\mathbb{R}^{n+1}$ since the image of a single basis element in $\mathbb{R}^n$ is all the basis elements of $\mathbb{R}^{n+1}$, satisfying (2).

Requirement (3) is a little more involved. The permutation representation decomposes as the direct sum of the trivial representation (which comes from the partition $(n)$) and the standard representation (which arises from the partition $(n-1,1)$). This means that the padded partitions for these are $\emptyset$ and $(1)$, respectively. Since precisely one copy of $V(\emptyset)$ and $V((1))$ exist in each representation, we have that (3) is satisfied.

Therefore, $(S_n, \mathbb{R}^n, i_n)$ is uniformly stable.

1.3. **Topology.** Topologists are interested in *invariants* of spaces – quantities or descriptors that stay constant when a space undergoes a homeomorphism. Many of these invariants are algebraic, involving concepts like homotopy groups, homology, or cohomology. In the presentation, we focus on cohomology (without defining it completely).

**Remark 3.** *Normally, we think of cohomology as pertaining to some space $X$, but in the presentation, I talk about the cohomology of the braid group. This is because to each group, we can assign a unique space (up to homotopy[2]). More details can be found in Section 2, and we need not concern ourselves with the details now.*

We now introduce the braid group (on $n$ strands), denoted $B_n$, and the pure braid group (on $n$ strands), denoted $P_n$ or $PB_n$. A more rigorous definition can be found in Section 3.

There is an action of $S_n$ on $P_n$ of renaming/permuting the marked points. This action extends to an $S_n$-action on $Hom(P_n, \mathbb{Q})$, or the group of homomorphisms from the pure braid group on $n$ strands to the rationals. We define the action as such: let $\sigma \in S_n$, $h \in Hom(P_n, \mathbb{Q})$, and $b \in P_n$. Then,

$$\sigma \cdot h(b) := h(\sigma^{-1} \cdot b).$$

It so happens that the set of homomorphisms is not only a group; it is a vector space with coefficients in $\mathbb{Q}$.

Therefore, we have an $S_n-$representation induced by the action of $S_n$ on $Hom(P_n, \mathbb{Q})$. This representation is uniformly stable!

To generalize this, let $H^1(P_n; \mathbb{Q}) = Hom(P_n, \mathbb{Q})$. This is the first cohomology [3] of $P_n$; we can also define a second cohomology, third cohomology, etc., and each of these is also a vector space on which $S_n$ acts. Then, we have the following thoerem of Church and Farb [1]:

**Theorem 4.** *For a fixed $i \geq 0$, the sequence of $S_n-$representations $\{H^i(P_n; \mathbb{Q})\}_n$ is uniformly stable, and it stabilizes when $n \geq 4i$.*

We note that the maps $\phi_n$ arise from algebraic topology as the maps between cohomology groups.

This result is surprising; it had previously been thought that this sequence is *not at all stable!* Furthermore, this result and associated examples were developed with the aid of a computer program.

Since the publication of this theorem, many people have built on Church and Farb's work. Representation stability has recently been applied to nested graphs with $S_n-$actions [6], homological stability of moduli spaces [5], and many, many other subjects.

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[2]Intuitively, homotopy is continuous deformation of a space.

[3]Generally, cohomology is a way to assign groups to spaces in such a way that we encode properties of a space. The cohomology group(s) are groups of homomorphisms from simplices in our space (up to some equivalences) to $\mathbb{Z}$.

## 2. On the equivalence of groups and spaces

We mention earlier that there is some form of equivalence between topological spaces and groups. More explicitly, this relationship is seen through $K(G, n)$ spaces. Given a group $G$ and a positive integer $n$, we can associate a space, $X$, such that $\pi_n(X) = G$.

For completeness, we include the definition of the $n^{\text{th}}$ homotopy group:

**Definition 5** ($n^{\text{th}}$ homotopy group)**.** *Given a topological space $X$, the $n^{th}$ homotopy group of $X$ is:*

$$\pi_n(X, x_0) = \{\textit{basepoint preserving maps } \phi : (S^n, s_0) \to (X, x_0) \mid \phi(s_0) = x_0\}/\{\textit{homotopy}\}$$

The homotopy group we are most interested in is $\pi_1(X, x_0)$, otherwise known as the **fundamental group**, which is the space of homotopy equivalence classes of closed loops in $X$.

**Example 6.** *Let $G = \mathbb{Z}$. Then, $K(\mathbb{Z}, 1)$ space is the unit circle $S^1$, since $\pi_1(S^1) = \mathbb{Z}$.*

*Why is $\pi_1(S^1, x_0) \cong \mathbb{Z}$? The formal proof is long, but the idea isn't too difficult. Loops in $S^1$ are just circles wrapping around the original circle itself – we can only wrap an integral number of times, and in either direction (thus covering both the negative integers and the positive integers).*

## 3. More on braid groups

We start by defining the braid group.

**Braid Group.** [3] Let $p_1, p_2, \ldots, p_n$ be $n$ marked points in $\mathbb{C}$. A *braid* is a collection of $n$ paths $f_i : [0, 1] \to \mathbb{C} \times [0, 1]$, $1 \leq i \leq n$, which we call *strands*, and a permutation $\sigma$ of $[n]$. The $f_i$ must satisfy the following properties:

- All strands are disjoint; i.e., the images of all $f_i$ are disjoint
- $f_i(0) = p_i$
- $f_i(1) = p_{\sigma(i)}$, and
- $f_i(t) \in \mathbb{C} \times t$.

This is not a very intuitive definition of a relatively familiar concept. Two helpful ways to think of braids are:

(1) Given a cylinder, we mark $n$ points on the top and the same $n$ points on the bottom. Then, we form the strands as paths in the cylinder starting at one of the points on the top of the cylinder and going down to a marked point on the bottom of the cylinder (without doubling back and without intersecting).

(2) We can visualize an element of $S_n$, the symmetric group on $n$ elements, as a diagram with the set of elements listed twice (one on the left and one on the right) and the permutation represented as lines going across. Then, at each crossing, we designate one of the lines to be on top and the other on the bottom. This is a good intuitive understanding of what a braid is. An example of a braid can be found in Figure 2.

The *braid group* on $n$ strands, denoted $B_n$, is the set of $n-$stranded braids with the operation of concatenation.

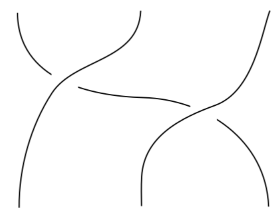

FIGURE 2. A braid with $n = 3$.

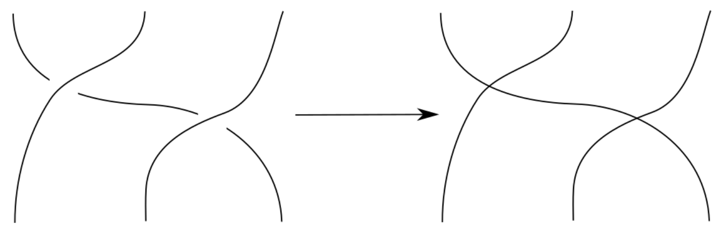

FIGURE 3. The effect of homomorphism $\varphi$ on the braid from Figure 2.

3.1. **Pure braid group.** Consider the map $\varphi$ taking the braid group $B_n$ to $S_n$ by forgetting the over/under strands in the image, as in Figure 3. It is not hard to see that this map is actually a surjective homomorphism, and its kernel is all braids taking the marked point $p_i$ to itself.

We denote the pure braid group on $n$ strands by $P_n$ or $PB_n$. An example of a pure braid on 3 strands can be seen in Figure 4.

FIGURE 4. A pure braid on 3 strands.

The pure braid group is important because it is the fundamental group of another space, called the *configuration space*, which is $X_n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for all } i \neq j\}$. Many people study configuration spaces of not only $\mathbb{C}^n$, but also of other manifolds, and the braid group is key to understanding how those spaces work.

## REFERENCES

[1] Thomas Church and Benson Farb. *Representation theory and homological stability.* Advances in Mathematics. 250-314, 2013.

[2] Thomas Church, Jordan Ellenberg, and Benson Farb. *Representation stability in cohomology and asymptotics for families of varieties over finite fields.* Contemporary Mathematics 620 (2014), 1–54.

[3] Benson Farb and Dan Margalit. *Primer on Mapping Class Groups.* Volume 49 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 2012. ISBN 9780691147949.

[4] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002

[5] Manuel Krannich. *Homological stability of topological moduli spaces.* Geometry & Topology 23 (2019). 2397–2474.

[6] Eric Ramos and Graham White. *Families of nested graphs with compatible symmetric-group actions.* Selecta Mathematica (2019). 25 (5), 70.