# THE ALEXANDER POLYNOMIAL

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Abstract. In this expository paper, we define the Alexander polynomial of a knot and look at an interesting result of de Rham relating certain representations of the knot group to the roots of the Alexander polynomial.

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## 1. Introduction

Discovered by James Wadell Alexander II in 1928, the Alexander polynomial is a knot invariant that encodes certain topological information about the knot complement. The purpose of this expository paper is to give a brief introduction to the Alexander polynomial and some of its properties. The paper is organized as follows. In section 3 we consider the homology of the infinite cyclic cover of the knot complement and use it to define the Alexander module and Alexander polynomial. Using Seifert surfaces, we also give some of their properties, including the famous skein relation. Then in section 4, we give an alternate definition of the Alexander module solely in terms of the knot group, from which the result of de Rham follows easily.

## 2. Preliminaries

In this section we review some basic notions from knot theory and algebra that will be used in the paper.

2.1. **Knots.** A (tame) *knot* is a subset of $S^3$ which is a piecewise-linear simple closed curve. Two knots $K_1$ and $K_2$ are *equivalent* if there is an ambient isotopy between them, i.e. a continuous family of homeomorphisms $F_t : S^3 \to S^3$, $t \in [0, 1]$, such that $F_0 = \mathrm{id}$ and $F_1(K_1) = K_2$. By results in topology, this turns out to be equivalent to saying that there exists a single orientation-preserving homeomorphism $F : S^3 \to S^3$ such that $F(K_1) = K_2$.

Note that if two knots $K_1$ and $K_2$ are equivalent, then certainly their complements $S^3 \setminus K_1$ and $S^3 \setminus K_2$ are homeomorphic 3-manifolds. One might ask whether the converse also holds, but it turns to be not quite true. In order to give a counterexample, let us first define the *mirror image* of a knot to be its image under any orientation-reversing homeomorphism $h : S^3 \to S^3$. (Observe that up to equivalence of knots, the resulting knot is independent of the choice of $h$.) More concretely, given a *diagram* of a knot (i.e. a projection of the knot onto some plane with over and under crossing indicated), a diagram of its mirror image is given by changing all the over-passes to under-passes. A simple counterexample to the above proposed converse is then provided by a trefoil knot and its mirror image: they have homeomorphic complements, but one can show that they are not equivalent, e.g. by comparing their Jones polynomials.

However, the converse turns out to be true if we use a slightly coarser notion of knot equivalence obtained by taking the original definition and declaring additionally that a knot and its mirror image are equivalent. (More simply stated, this weaker definition states that two knots $K_1$ and $K_2$ are equivalent if there exists any homeomorphism $F : S^3 \to S^3$ (not necessarily orientation-preserving) such that $F(K_1) = K_2$.) This is the Gordon-Luecke Theorem (1989).

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The upshot of all this is that the task of distinguishing knots is (essentially) equivalent to the task of distinguishing their 3-manifold complement spaces.

Given a knot $K$, it is thus natural to consider algebraic invariants of its complement. For instance, one can consider the fundamental group of the complement, which is called the *knot group* of $K$. Given a diagram of a knot $K$, it turns out that there is a simple algorithm for writing down a presentation of its knot group; the resulting presentation is called a *Wirtinger presentation* for the knot group of $K$. To begin, let there be $n$ arcs in the knot diagram, and associate to the $i^{\text{th}}$ arc of the diagram the symbol $a_i$. (The *arcs* of a knot diagram are just its connected components. For instance, the knot diagram of the figure-eight knot in Figure 2 has four arcs, labeled $a_1, a_2, a_3, a_4$.) The symbols $a_1, a_2, \ldots, a_n$ will be the generators of the knot group. Now give the knot an orientation, and at each crossing $c$ define the relation $r_c$ described in Figure 1; it will depend on whether the crossing is a "positive" crossing (left) or "negative" crossing (right). Then it turns out that the knot group has the presentation

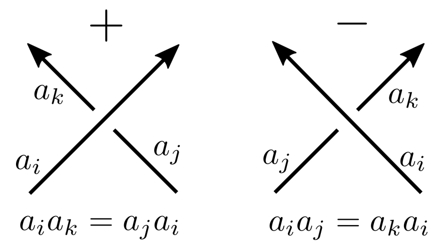


Figure 1. Wirtinger relations for positive and negative crossings.

$\langle a_1, a_2, \ldots, a_n \,|\, r_1, r_2, \ldots r_n \rangle$. Each $a_i$ may be interpreted as a loop that starts at a fixed basepoint somewhere above the page, travels directly to the $i^{\text{th}}$ arc, encircles it once in "right-hand rule" fashion, and finally returns directly to the basepoint. It is then not hard to see that the relations $r_1, r_2 \ldots, r_n$ must hold. More difficult is showing that these are a complete set of relations. One proof of this fact involves constructing a suitable 2-dimensional CW complex that the knot complement deformation retracts onto, and then using van Kampen's theorem to calculate the fundamental group of the complex. More details can be found in [1, p. 55]. Finally, one can show that each of the relations is a consequence of the others, so any one of them may be dropped.

**Example 2.1.** Figure 2 shows a diagram of the figure-eight knot, with arcs $a_1, a_2, a_3, a_4$ labeled and orientation as shown. Using the algorithm just described, we find that one presentation

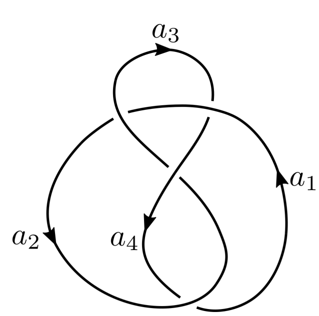


Figure 2. The figure-eight knot.

for the knot group of the figure-eight knot is

$$\langle a_1, a_2, a_3, a_4 \,|\, a_3a_2 = a_1a_3,\ a_2a_4 = a_1a_2,\ a_4a_2 = a_3a_4,\ a_1a_4 = a_3a_1\rangle.$$

As noted above, any one of the relations can be dropped.

By abelianizing the knot group and using the fact that the $a_i$ are all conjugate to each other, we see that the first homology group $H_1(S^3 \setminus K)$ of the knot complement is always $\mathbb{Z}$. (Note: in this paper, all homology groups are with $\mathbb{Z}$ coefficients.) Moreover, we see that it is generated by the homology class corresponding to any one of the $a_i$, or in other words the homology class of a meridian curve of a small closed solid-torus tubular neighborhood of the knot. If the knot $K$ is oriented, then it is customary to think of $1 \in \mathbb{Z}$ as representing the homology class of such a meridian curve, oriented so that it encircles the oriented knot $K$ in right hand rule fashion (as opposed to the same curve with the opposite orientation).

Now if $C$ is any oriented simple closed curve in $S^3 \setminus K$, the homology class of $[C] \in H_1(S^3 \setminus K) = \mathbb{Z}$ is called the *linking number* of the oriented knots $C$ and $K$, denoted $\mathrm{lk}(C, K)$. Given a diagram of $C$ and $K$, their linking number can be though of more concretely as half the sum of the signs of the crossings where one strand is from $C$ and the other is from $K$ (exercise). By definition, the sign of a crossing is $+1$ if the crossing is positive in the sense of Figure 1, and $-1$ if the crossing is negative. It follows in particular from this that $\mathrm{lk}(C, K) = \mathrm{lk}(K, C)$.

2.2. **Finitely-presented modules.** Let $M$ be a module over a commutative ring $A$. A *finite presentation* for $M$ is a finite set of generators $x_1, \dots x_n \in M$ along with a finite complete set of relations $r_1, \dots r_m$ among them. This means that $M$ is isomorphic to the quotient of the free $A$-module $\langle x_1, \dots x_n\rangle$ by the submodule generated by the relations $r_1, \dots r_m$. In other words, a finite presentation for $M$ is an exact sequence of $A$-modules

$$A^m \xrightarrow{\alpha} A^n \xrightarrow{\varphi} M \to 0.$$

The images of the standard basis elements of $A^n$ under $\varphi$ correspond to the generators $x_i$ from above and the images of the standard basis elements of $A^m$ under $\varphi \circ \alpha$ correspod to the relations $r_i$. If such a finite presentation for $M$ exists, then we say that $M$ is *finitely presented*, and we can summarize the presentation by the matrix representation of the map $\alpha$ with respect to the standard bases of $A^m$ and $A^n$. This $n \times m$ matrix is called a *presentation matrix* for $M$. In other words, writing $r_i = a_{1i}x_1 + \cdots + a_{ni}x_n$ for all $1 \le i \le n$, the presentation matrix corresponding to this presentation is simply the matrix $(a_{ij})_{1\le i\le n, 1\le j\le m}$; the rows correspond to the generators and the columns correspond to the relations among the generators.

Next we state a fundamental fact about finite-presented modules. A short proof can be found in [2, p. 49–50].

**Proposition 2.2.** *Any two presentation matrices $P$ and $P'$ of a module $M$ are related by a sequence of the following types of moves and their inverses:*

*(a) Permutation of rows or columns*

*(b) Replacement of the matrix $P$ by* $\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$

*(c) Addition of an extra column of zeros*

*(d) Addition of a scalar multiple of a row (or column) to another row (or column)*

Because modules can be rather complicated objects, it is sometimes useful to associate to them simpler invariants such as the following.

**Definition 2.3.** Let $M$ be a finitely-presented module over a commutative ring $A$. The $i^{\text{th}}$ *elementary ideal* $M$ is the ideal of $A$ generated by the $(n - i + 1) \times (n - i + 1)$ minors in any $n \times m$ presentation matrix of $M$.

Recall that a $k \times k$ minor of a matrix $P$ is the determinant of some $k \times k$ submatrix of $P$, i.e. a $k \times k$ matrix formed from $k$ rows and $k$ columns of $M$. The reader might like to verify using Proposition 2.2 that this definition does not depend on the choice of presentation matrix, and so it is well-defined.

## 3. The infinite cyclic cover of the knot complement

3.1. **Alexander module and Alexander polynomial.** We have seen that the knot group is a knot invariant. However, in general it is difficult to distinguish groups from their presentations, so it is desirable to obtain a simpler invariant. The first homology group of the knot complement is unfortunately too simple, since it is always $\mathbb{Z}$. However, it turns out that considering the first homology group of the infinite cyclic cover of the complement will lead to the fairly good polynomial invariant known as the Alexander polynomial.

Let $K$ be a knot, $X$ be its complement in $S^3$, and $G = \pi_1(X)$ be its knot group. For any group $H$, let $H'$ denote its commutator subgroup. Let $p : X_\infty \to X$ be the covering space corresponding to the normal subgroup $G' \leq G = \pi_1(X)$, so that $p_*(\pi_1(X_\infty)) = G'$. Recall from covering space theory that the group of deck transformations is isomorphic to $G/G' = H_1(X) = \mathbb{Z}$, so we refer to $X_\infty$ as the *infinite cyclic cover* of $X$. Observe that $X_\infty$ is the *only* covering space with group of deck transformations $\mathbb{Z}$ (exercise), which justifies our referring to $X_\infty$ as *the* infinite cyclic cover.

Denote by $t : X_\infty \to X_\infty$ one of the two generators for the group $\mathbb{Z}$ of deck transformations of $X_\infty$. Then $t$ induces an isomorphism $t_* : H_1(X_\infty) \to H_1(X_\infty)$ on homology, which by abuse of notation we will denote simply by $t$. This gives rise to a group action of $\langle t \rangle$ on $H_1(X_\infty)$, which in turn gives rise in the natural way to a $\mathbb{Z}[t, t^{-1}]$-module structure on $H_1(X_\infty)$. This module is of course a knot invariant. Now we are ready to define the Alexander polynomial. Actually, there is a whole family of Alexander polynomials.

**Definition 3.1.** The *Alexander module* of a knot $K$ is the $\mathbb{Z}[t, t^{-1}]$-module $H_1(X_\infty)$. It turns out that the Alexander module is finitely-presented, and its $i^{\text{th}}$ elementary ideal is called the $i^{\text{th}}$ *Alexander ideal* of $K$. The $i^{\text{th}}$ *Alexander polynomial* of $K$ is a generator of the smallest principal ideal of $\mathbb{Z}[t, t^{-1}]$ containing the $i^{\text{th}}$ Alexander ideal. The $1^{\text{st}}$ Alexander polynomial of $K$ is called simply the *Alexander polynomial*, and is denoted $\Delta_K(t)$.

Note that Alexander polynomials are defined only up to multiplication by the units $\pm t^n$ ($n \in \mathbb{Z}$) of the ring $\mathbb{Z}[t, t^{-1}]$. In this paper we will focus only on *the* Alexander polynomial, meaning the $1^{\text{st}}$ Alexander polynomial. Definition 3.1 means in particular that if we happen to have a square presentation matrix for the Alexander module, then the Alexander polynomial is simply the determinant of that square matrix.

3.2. **Seifert surfaces.** In order to be able to prove things about the Alexander polynomial, it will be useful to understand the infinite cyclic cover $X_\infty$ in a more geometric way. For this we need the concept of a Seifert surface of a knot.

**Definition 3.2.** A *Seifert surface* of a knot $K$ is a connected compact orientable surface in $S^3$ whose boundary is $K$. The *genus* of $K$ is the minimum genus of any Seifert surface for $K$.

Given a diagram of a knot, there is an algorithm called *Seifert's algorithm* that contructs a Seifert surface for the knot. For a description of this algorithm, see for instance [2, p. 16]. In particular, every knot has a Seifert surface.

Let $K$ be a knot with Seifert surface $F$. It is convenient to think of the knot complement $X$ as the closure of the complement of some tubular neighborhood $N$ of $K$. Since $F \cap X$ is just $F$ with a small neighborhood of $\partial F$ removed, we'll refer to $F \cap X$ as $F$ from now on. Since $F$ is orientable, it has a neighborhood $F \times [-1, 1]$ in $X$, where $F \times \{0\}$ is identified with $F$. Define maps $p^-, p^+ : F \to S^3 \setminus F$ by $p^-(x) = (x, -1)$ and $p^+(x) = (x, 1)$. This maps "push" points slightly off of $F$ into $S^3 \setminus F$.

To get a glimpse of how Seifert surfaces can help us understand the infinite cycic cover $X_\infty$ of $X$, consider cutting $X$ along $F$. In other words, consider the space $Y$ obtained by removing from $X$ the neighborhood $F \times (-1, 1)$ of $F$ from before. Note that $p^-(F)$ and $p^+(F)$ are part of the boundary of $Y$, and $X$ can be recovered from $Y$ by just gluing togeher $p^-(F)$ and $p^+(F)$ in the natural way. Then the infinite cyclic cover $X_\infty$ of $X$ can be contructed from gluing together countably many copies of $Y$ in linear fashion. (Think of how $\mathbb{R}$ is the infinite cyclic cover of $S^1$.)

**Example 3.3.** If $K$ is the unknot, then it has a Seifert surface $F$ which is just a disc $D^2$. Cutting $X$ along $F$ is like removing an open ball from $S^3$ and results in $Y = D^2 \times [-1, 1]$, with $p^-(F) = D^2 \times \{-1\}$ and $p^+(F) = D^2 \times \{1\}$. Gluing together countably many copies of $Y$ together then yields $X_\infty = D^2 \times \mathbb{R}$. Since this is contractible, the Alexander module is just $H_1(X_\infty) = 0$. A presentation matrix for this module is simply $(1)$, and the Alexander polynomial is the determinant of this matrix, which is 1.

Unfortunately, for more complicated knots it is not really possible to proceed directly like this. However, using this description of $X_\infty$ it is still possible to calculate the Alexander module $H_1(X_\infty)$ in terms of the Seifert surface (Theorem 3.5). But before we can do so, we need to first introduce a few more definitions.

**Definition 3.4.** Let $K$ be a knot with Seifert surface $F$. The corresponding *Seifert form* is the unique biliner map

$$\alpha : H_1(F) \times H_1(F) \to \mathbb{Z}$$

satisfying $\alpha([x], [y]) = \mathrm{lk}(p^-(x), y) = \mathrm{lk}(x, p^+(y))$ for all oriented simple closed curves $x$ and $y$ in $F$.

Recall that $\mathrm{lk}(\cdot, \cdot)$ denotes linking number. A bit of algebraic topology shows that there is indeed a unique such bilinear map [2, p. 51-53]. Given a basis for $H_1(F)$, the matrix representation of the Seifert bilinear form is called a *Seifert matrix*. Explicitly, if $\{f_i\}$ are oriented simple closed curves in $F$ such that $\{[f_i]\}$ is a basis for $H_1(F)$, then the entries of the Seifert matrix $A$ are given by $A_{ij} = \alpha([f_i], [f_j]) = \mathrm{lk}(p^-(f_i), f_j)$.

Now with the above description of $X_\infty$ in terms of $Y$, a little algebraic topology calculation shows how the Alexander module $H_1(X_\infty)$ is related to a Seifert matrix ([2, p. 54-55]):

**Theorem 3.5.** *Let $K$ be a knot with Seifert surface $F$ and corresponding Seifert form $\alpha$. If $A$ is a matrix representing $\alpha$ with respect to any basis of $H_1(F)$, then $tA - A^T$ is a presentation matrix for the Alexander module $H_1(X_\infty)$ of $K$. Since this is a square matrix, the Alexander polynomial is its determinant* $\det(tA - A^T)$.

This theorem is important because it allows us to prove many interesting properties about the Alexander polynomial, such as the following.

**Corollary 3.6.** *For any knot $K$,*

$$\mathit{genus}(K) \geq \frac{1}{2} \cdot \mathit{breadth}(\Delta_K(t)),$$

*where the breadth of a Laurent polynomial is defined as the difference between the highest and lowest degrees appearing in the Laurent polynomial.*

*Proof.* Let $g$ denote the genus of $K$, and $A$ denote the Seifert matrix corresponding to the associated Seifert surface. Then $A$ is a $2g \times 2g$ matrix, so $\Delta_K(t) = \det(tA - A^T)$ is a bona fide polynomial with degree (and hence breadth) at most $2g$. The proof is complete. $\square$

**Corollary 3.7.** *For any knot $K$,*

*(1)* $\Delta_K(t) = \Delta_K(t^{-1})$ *up to multiplication by units.*
*(2)* $\Delta_K(1) = \pm 1$.
*(3)* $\Delta_K(t) = a_0 + a_1(t+t^{-1}) + a_2(t^2+t^{-2}) + \cdots$ *up to multiplication by units, where* $a_i \in \mathbb{Z}$ *and* $a_0$ *is odd.*

*Proof.* These facts follow easily from Theorem 3.5. See [2, p. 58] for details. □

It should be noted that much of what has been said up to this point applies not only to knots but more generally to oriented links. (A *link* is a disjoint collection of knots.) In particular we can define an Alexander polynomial for them, and Theorem 3.5 still holds.

Recall that the Alexander polynomial is defined only up to multiplication by the units $\pm t^n$ of $\mathbb{Z}[t, t^{-1}]$. In order to state the next result, we need a normalized form of the Alexander polynomial.

**Definition 3.8.** For any oriented link $L$, its *Alexander-Conway polynomial* is defined as

$$\Delta_L(t) := \det(t^{1/2}A - t^{-1/2}A^T) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$$

where $A$ is any Seifert matrix for $L$. Thus if $A$ is an $n \times n$ matrix, then $\Delta_L(t) = t^{-n/2}\det(tA - A^T)$.

Although it's not obvious, this definition is well-defined and an invariant.

The following relation allows one to compute Alexander polynomials recursively.

**Corollary 3.9** (Skein relation)**.** *Let $L_+$, $L_-$, and $L_0$ be oriented links which differ only in some small ball, where they are as shown in Figure 3. Then their Alexander-Conway polynomials satisfy the relation*

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{-1/2} - t^{1/2})\Delta_{L_0}(t).$$

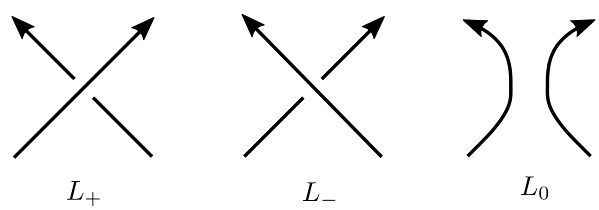


FIGURE 3. Three oriented links differing only in a small ball.

*Proof.* Use Theorem 3.5. See [2, p. 82-83] for details. □

## 4. The knot group

4.1. **Alexander module in terms of the knot group.** In this section we give another, equivalent definition of the Alexander module of a knot in terms of just its knot group (Proposition 4.2). The idea is to take the homological Definition 3.1 and reformulate everything in terms of fundamental groups using the fact that the first homology group of a space is just the abelianization of its fundamental group.

Let us first recall the notation. As before, let $K$ be a knot, $X$ be its complement in $S^3$, $G = \pi_1(X, x_0)$ be its knot group, and $p : (X_\infty, \widetilde{x}_0) \to (X, x_0)$ be the infinite cyclic covering space of $X$, so that $p_*(\pi_1(X_\infty, \widetilde{x}_0)) = G'$. Also, let $t : X_\infty \to X_\infty$ be one of the two generators for the group $\mathbb{Z}$ of deck transformations of $X_\infty$.

Since $p_*$ is injective, we have that the map $p_* : \pi_1(X_\infty, \widetilde{x}_0) \to G'$ is an isomorphism of groups. We can therefore identify the Alexander module $H_1(X_\infty)$ with the abelian group $\pi_1(X_\infty, \widetilde{x}_0)/\pi_1(X_\infty, \widetilde{x}_0)' \cong G'/G''$ in the following natural way (where $G'' = (G')'$ denotes the commutator subgroup of $G'$). First let $h : \pi_1(X_\infty, \widetilde{x}_0) \to H_1(X_\infty)$ denote the natural map that sends $[f]$ to the homology class of $f$, when interpreted as a 1-cycle. Then it is well-known that $h$ is a surjective homomorphism and that its kernel is the commutator subgroup of $\pi_1(X_\infty, \widetilde{x}_0)$. (This is of course the reason why the first homology group is the abelianization of the fundamental group. For a proof of these facts, see for example [1, p. 166].) Hence the group isomorphism $p_* : \pi_1(X_\infty, \widetilde{x}_0) \to G'$ induces a group isomorphism $\varphi : H_1(X_\infty) \to G'/G''$ defined by the following commutative diagram (where $\mathrm{ab} : G' \to G'/G''$ denotes the abelianization quotient map):

$$\begin{tikzcd}
\pi_1(X_\infty, \widetilde{x}_0) \arrow[r, "p_*"] \arrow[d, "h"] & G' \arrow[d, "\mathrm{ab}"] \\
H_1(X_\infty) \arrow[r, "\varphi"] & G'/G''
\end{tikzcd}$$

Note that $G'/G''$ is currently only an abelian group and $\varphi$ is only a *group* isomorphism. We would like now to define a $\mathbb{Z}[t, t^{-1}]$-module structure on $G'/G''$ so that $\varphi$ is not only a group isomorphism but also a $\mathbb{Z}[t, t^{-1}]$-module isomorphism. This will give us a new definition of the Alexander module as the $\mathbb{Z}[t, t^{-1}]$-module $G'/G''$.

To define the $\mathbb{Z}[t, t^{-1}]$-module structure on $G'/G''$, it suffices to define how $t \in \mathbb{Z}[t, t^{-1}]$ acts on $G'/G''$, and we want this to correspond to the action of $t_*$ on the Alexander module $H_1(X_\infty)$. There is only once choice here, and it is the unique map $t' : G'/G'' \to G'/G''$ that makes the following diagram commute:

$$\begin{tikzcd}
H_1(X_\infty) \arrow[r, "t_*"] \arrow[d, "\varphi"] & H_1(X_\infty) \arrow[d, "\varphi"] \\
G'/G'' \arrow[r, "t'"] & G'/G''
\end{tikzcd}$$

We claim that we can describe $t'$ in another, more explicit way as follows.

**Lemma 4.1.** *Let $g \in G$ denote an element that becomes a generator in the abelianization $G/G' \cong \mathbb{Z}$. Then the map $t' : G'/G'' \to G'/G''$ defined by conjugation by $g$, i.e.*

$$t'([x]) = [gxg^{-1}],$$

*satisfies the above commutative diagram.*

*Proof.* We first show how $g$ and $t_*$ are related. Note that $g$ is the homotopy class of some loop $\gamma$ based at $x_0$. Denote by $\widetilde{\gamma}$ the lift of $\gamma$ to $X_\infty$ beginning at $\widetilde{x}_0$. Then the unique deck transformation of $X_\infty$ that sends $\widetilde{x}_0 = \widetilde{\gamma}(0)$ to $\widetilde{x}_1 := \widetilde{\gamma}(1)$ is a generator for the group $\mathbb{Z}$ of deck transformations, as can be seen from the definition of $g$ and the standard isomorphism between the group of deck transformations and $\pi_1(X)/p_*(\pi_1(X_\infty)) = G/G'$. (For a description of this isomorphism, see for example [1, p. 71].) This deck transformation is either $t$ or $t^{-1}$, but we can assume that it is $t$ by replacing $g$ by $g^{-1}$ if necessary.

Now to actually prove the lemma, let $[[\ell]] \in G'/G''$, where $\ell$ is a loop in $X$ based at $x_0$. We want to show that $t'([[\ell]]) = \varphi(t_*(\varphi^{-1}([[\ell]])))$. From the first commutative diagram, we get that $\varphi^{-1}([[\ell]])$ is the homology class of the lifted loop $\widetilde{\ell}$ of $\ell$ beginning at $\widetilde{x}_0$. Applying $t_*$ to this gives the homology class of $t \circ \widetilde{\ell}$, which is a loop based at $t(\widetilde{x}_0) = \widetilde{x}_1$. Now recall that $\widetilde{\gamma}$ is a path from $\widetilde{x}_0$ to $\widetilde{x}_1$, so the loops $t \circ \widetilde{\ell}$ and $\widetilde{\gamma} \cdot (t \circ \widetilde{\ell}) \cdot \widetilde{\gamma}^{-1}$ are freely homotopic and thus homologous. Finally, applying $\varphi$ to the homology class of $\widetilde{\gamma} \cdot (t \circ \widetilde{\ell}) \cdot \widetilde{\gamma}^{-1}$ (which is a loop based at $\widetilde{x}_0$) and using the first commutative diagram yields $[[\gamma \cdot \ell \cdot \gamma^{-1}]] = [g[\ell]g^{-1}] = t'([[\ell]])$ as desired. This establishes the lemma. $\square$

In summary, we have shown:

**Proposition 4.2.** *Let $K$ be a knot, $G$ be its knot group, and $g \in G$ an element that becomes a generator in the abelianization $G/G' \cong \mathbb{Z}$. Then the $\mathbb{Z}[t, t^{-1}]$-module $G'/G''$ defined by $t \cdot [x] = [gxg^{-1}]$ is isomorphic to the Alexander module $H_1(X_\infty)$ of $K$.*

Of course, to calculate the Alexander polynomial from the Alexander module we need to be able to write down a presentation matrix for the Alexander module, which we will see how to do next.

Let $G = \langle a_1, a_2, \ldots a_n \,|\, r_1, r_2, \ldots, r_{n-1} \rangle$ be a Wirtinger presentation of the knot group. Because the $a_i$ are all conjugate to each other (this follows directly from the relations), the images of $a_1, a_2, \ldots, a_n$ in the abelianization $G/G' \cong \mathbb{Z}$ are all the same, namely one of the generators 1 or $-1$. If we now define

$$b_i := a_i a_n^{-1}, \quad \text{for } i = 1, 2, \ldots, n,$$

then it follows from the previous sentence that $b_i \in G'$ for all $i = 1, \ldots n-1$. (Note that $b_n = e$.) Moreover, it's an easy exercise to show that the elements $b_1, b_2, \ldots, b_{n-1} \in G'$ together with their conjugates $a_n b_1 a_n^{-1}, a_n b_2 a_n^{-1}, \ldots, a_n b_{n-1} a_n^{-1} \in G'$ by $a_n$ generate $G'$. (Use the fact that $G'$ is generated by the commutators among only the $a_i$.) For convenience, define

$$b_i' := a_n b_i a_n^{-1} \in G', \quad \text{for } i = 1, 2, \ldots, n,$$

so the previous statement is that $b_1, b_2, \ldots, b_{n-1}$, $b_1', b_2', \ldots, b_{n-1}' \in G'$ generate $G'$. (Note that $b_n' = e$.)

Of course, these generators of $G'$ satisfy several relations. Namely, given a relation from the Wirtinger presentation of $G$ of the form $a_i a_k = a_j a_i$, i.e.,

$$a_i a_k a_i^{-1} a_j^{-1} = e, \tag{4.1}$$

we can substitute $a_i = b_i a_n$ for all $i$ into this to get $(b_i a_n)(b_k a_n)(a_n^{-1} b_i^{-1})(a_n^{-1} b_j^{-1}) = e$, which can be rewritten as $b_i(a_n b_k a_n^{-1})(a_n b_i^{-1} a_n^{-1}) b_j^{-1} = e$, i.e.

$$b_i b_k' (b_i')^{-1} b_j^{-1} = e. \tag{4.2}$$

Thus each of the $n-1$ relations of the form (4.1) in the Wirtinger presentation of $G$ gives rise to a corresponding relation (4.2) among the generators $b_1, \ldots, b_{n-1}$, $b_1', \ldots, b_{n-1}'$ of $G'$.

Now we pass from $G'$ to its abelianization $G'/G''$. The abelianization is generated (as a group) by the images of the generators $b_1, \ldots, b_{n-1}$, $b_1', \ldots, b_{n-1}'$ of $G'$. So for $i = 1, 2, \ldots, n$, denote by $c_i \in G'/G''$ the image of $b_i$ in $G'/G''$. As for the images of the $b_i'$, recall from Proposition 4.2 that $G'/G''$ has a $\mathbb{Z}[t, t^{-1}]$-module structure, which we can use to write the image of $b_i' = a_n b_i a_n^{-1}$ in $G'/G''$ as $t \cdot c_i$. Therefore $G'/G''$ is generated as an abelian group by $c_1, \ldots, c_{n-1}$, $t \cdot c_1, \ldots, t \cdot c_{n-1}$. Hence as a $\mathbb{Z}[t, t^{-1}]$-module, $G'/G''$ is generated by $c_1, \ldots, c_{n-1}$. Furthermore, in the abelianization the relation $b_i b_k' (b_i')^{-1} b_j^{-1} = e$ from (4.2) turns into (using additive notation now because we are now in an abelian group) the relation $c_i + t \cdot c_k - t \cdot c_i - c_j = 0$, i.e.

$$(1 - t) \cdot c_i + t \cdot c_k - c_j = 0. \tag{4.3}$$

(Because $c_n = 0$, all appearances of $c_n$ can be dropped to make the above equation a relation among only $c_1, \ldots, c_{n-1}$.) In summary, $G'/G''$ is a $\mathbb{Z}[t, t^{-1}]$-module with generators $c_1, \ldots, c_{n-1}$ that satisfy $n-1$ relations of the above form (4.3), each coming from a relation in the Wirtinger presentation of $G$ of the form (4.1). Although we do not prove it here, it turns out that these

$n-1$ relations form a complete set of relations among the generators $c_1,\dots,c_{n-1}$ [3, p. 191], so this gives us an $(n-1)\times(n-1)$ presentation matrix for the Alexander module $G'/G''$.

**Example 4.3.** Recall from Example 2.1 that the knot group of the figure-eight knot is generated by $a_1,a_2,a_3,a_4$ with relations

$$\begin{aligned} a_3a_2 &= a_1a_3,\\ a_2a_4 &= a_1a_2,\\ a_4a_2 &= a_3a_4. \end{aligned}$$

Therefore the Alexander module of the figure-eight knot is generated by $c_1,c_2,c_3$ with relations

$$\begin{aligned} (1-t)\cdot c_3 + t\cdot c_2 - c_1 &= 0,\\ (1-t)\cdot c_2 - c_1 &= 0,\\ t\cdot c_2 - c_3 &= 0. \end{aligned}$$

So a presentation matrix for the Alexander module is $\begin{pmatrix} -1 & t & 1-t\\ -1 & 1-t & 0\\ 0 & t & -1\end{pmatrix}^T$. The Alexander polynomial is then the determinant of this matrix, which is $t^2-3t+1$.

4.2. **Representations of the knot group.** Let $\mathcal{L}$ denote the group of Möbius transformations of $\mathbb{C}$ of the form $z\mapsto xz+y$, for some $x,y\in\mathbb{C}$ with $x\neq 0$. For brevity, we denote the transformation $z\mapsto xz+y$ simply by $(x,y)$. Written explicitly, the group operation is $(x_1,y_1)\circ(x_2,y_2)=(x_1x_2, x_1y_2+y_1)$. In this section we will study representations (i.e. homomorphisms) from the knot group $G$ of a knot to $\mathcal{L}$.

*Remark* 4.4. Because $\mathcal{L}$ is ismorphic to the matrix groups

$$\mathcal{M}_1 := \left\{ \begin{pmatrix} x & y\\ 0 & 1\end{pmatrix} : x,y\in\mathbb{C}, x\neq 0\right\} \subset \mathrm{GL}_2(\mathbb{C})$$

and

$$\mathcal{M}_2 := \left\{ \begin{pmatrix} t & a\\ 0 & t^{-1}\end{pmatrix} : t,a\in\mathbb{C}, t\neq 0\right\} /\{\pm I\} \subset \mathrm{PSL}_2(\mathbb{C}),$$

(via the ismomorphisms $\begin{pmatrix} x & y\\ 0 & 1\end{pmatrix}\mapsto (x,y)$ and $\begin{pmatrix} t & a\\ 0 & t^{-1}\end{pmatrix}\mapsto (t^2, at)$, respectively), the following discussion can also be stated in terms of representations of $G$ into $\mathcal{M}_1$ or $\mathcal{M}_2$.

To begin, let $\rho: G\to\mathcal{L}$ be a representation and let $\langle a_1,a_2,\dots,a_n \,|\, r_1,r_2,\dots,r_{n-1}\rangle$ be a Wirtinger presentation of $G$. Denote $\rho(a_i)=(x_i,y_i)$. We start by making a series of small observations.

(1) Since the $a_i$ are all conjugate to each other, so are their images $\rho(a_i)$, which (since two transformations in $\mathcal{L}$ are conjugate if and only if they have the same scale factor) immediately forces all the $x_i$ to be equal to the same value, say, $x\in\mathbb{C}$.
(2) Observe that $\rho$ is conjugate/isomorphic to a representation with $y_n=0$ via the isomorphism $z\mapsto z+\frac{y_n}{1-x}$ (to come up with this isomorphism, think about fixed points). So we may assume without loss of generality that $y_n=0$ from now on. Note that the value of $x$ is unchanged by this (in fact, by any) isomorphism.
(3) Note if $y_i=0$ for all $i=1,\dots,n$, then the image of $\rho$ is the subgroup of $\mathcal{L}$ generated by $(x,0)$; hence $\rho$ is an *abelian* representation, in the sense that its image is an abelian group (in this case $\mathbb{Z}$). Conversely, it is easy to see that if $\rho$ is an abelian representation, then $y_i=0$ for all $i=1,\dots,n$. Because abelian representations factor through

$G/G' \cong \mathbb{Z}$, they are uninteresting, and so we will restrict our attention to non-abelian representations $\rho$ from now on.

Next, each relation from the presentation of $G$ is of the form $a_i a_k = a_j a_i$, which upon applying $\rho$ yields $(x, y_i) \circ (x, y_k) = (x, y_j) \circ (x, y_i)$. This simplifies to $(x, xy_k + y_i) = (x, xy_i + y_j)$, or equivalently

$$(1 - x)y_i + xy_k - y_j = 0.$$

This gives an $(n-1) \times (n-1)$ linear system in $y_1, y_2, \ldots, y_{n-1}$, represented by, say, the $(n-1) \times (n-1)$ matrix $M$. (Recall from point (2) above that $y_n = 0$.) Remarkably, this equation is exactly the same as equation (4.3) from the previous section if we identify each $y_i$ with $c_i$ and $x$ with $t$. Therefore $M^T$, with $x$ replaced by $t$, is in fact a presentation matrix for the Alexander module.

Finally we come to the denouement of the entire paper. Since $\rho$ is not abelian (see point (3) above), we do not have $y_i = 0$ for all $i$, which implies that $M$ is not invertible. Thus $\det M^T = \det M = 0$, which means that $x$ is a root of the determinant of the Alexander module, i.e. $x$ is a root of the Alexander polynomial. We can extend this result slightly with the following two observations: (i) $\rho$ maps the commutator subgroup $G'$ of $G$ to the commutator subgroup of $\mathcal{L}$, which is none other than the subgroup of translations $\{(1, y) : y \in \mathbb{C}\}$, and (ii) the set of roots of the Alexander polynomial is closed under inversion (by Corollary 3.7 (1)). We obtain the following result.

**Theorem 4.5.** *Let $K$ be a knot with knot group $G$ and $\rho : G \to \mathcal{L}$ be a non-abelian representation of $G$. Then for any $g \in G$ whose image in the abelianization $G/G' \cong \mathbb{Z}$ is a generator $\pm 1$, if we denote $\rho(g) = (x, y)$, then $x$ is in fact a root of the Alexander polynomial $\Delta_K(t)$.*

An amusing corollary is that if $\Delta_K(t) = 1$ (for instance, if $K$ is the unknot or the $(-3, 5, 7)$-pretzel knot), then no non-abelian representations $\rho : G \to \mathcal{L}$ exist.

## References


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