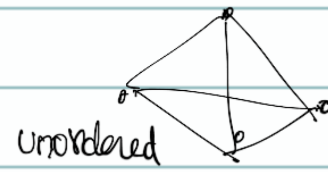
Random Graphs & Limits

1985 A. Frieze.

K_n – Complete graph



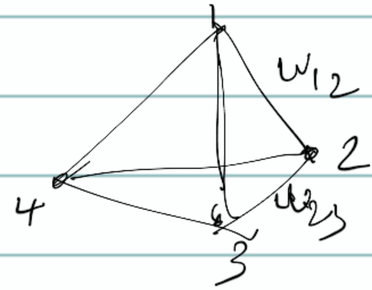
K_4

n vertices & all $\binom{n}{2}$ unordered pairs are edges.

On each edge, I put i.i.d. Uniform $[0,1)$ weight $\omega_{i,j}$.

$\omega_{i,j} = \omega_{j,i} \quad \forall\ i \neq j \in [n] = \{1, \ldots, n\}$

$\{ \omega_{i,j},\ 1 \leq i < j \leq n \}$ are i.i.d. $U(0,1)$ r.v.'s.



K_4

$G \subseteq K_4 \quad \omega(G) = \sum_{e \in G} \omega(e)$

$M_n^* = \min \{ \omega(G) : G \subseteq K_n \text{ \& } G \text{ is connected} \}$.

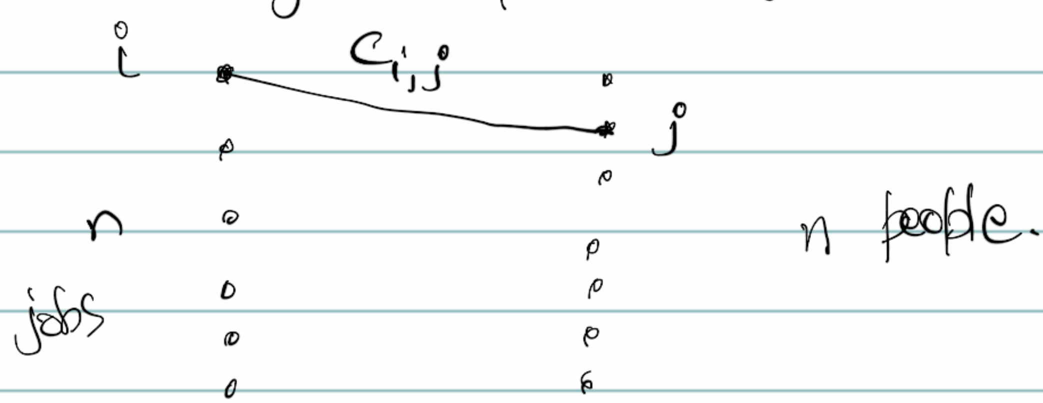
$\exists$ a subgraph G_n^* (called the minimal spanning tree)

$\ni \quad M_n^* = w(G_n^*)$.

Frieze's thm: $\quad \mathbb{E} M_n^* \longrightarrow \sum_{k=1}^{\infty} \frac{1}{k^3} = \zeta(3)$

Riemann Zeta fn.

Random Assignment Problem (1962).



$c_{i,j}$ = cost of assigning job i to person j.

Assume $c_{i,j}$ i.i.d. $\text{EXP}(1)$ r.v.

Job assignment - $\pi : [n] \to [n]$, a bijection.

$$C_n^* = \min_{\pi \in S_n} \sum_{i=1}^{n} c_{i,\pi(i)}.$$

$\hookrightarrow$ permutation group.

$\mathbb{E}[C_n^*] \leq C\log n$

$\downarrow$

$\leq C$

(1987) Mezard - Parisi prediction.

$$\mathbb{E}[C_n^*] \longrightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2).$$

Qn. from Measure-theoretic prob.

$X_{i,n}$, $1 \leq i \leq n$ i.i.d. $\mathrm{Ber}\left(\frac{\lambda}{n}\right)$ r.v. $n > \lambda$.

$\hookrightarrow$ Bernoulli

$$S_n = \sum_{i=1}^{n} X_{i,n} \stackrel{d}{=} \mathrm{Bin}\left(n, \frac{\lambda}{n}\right).$$

$f : \mathbb{R} \to \mathbb{R}$ be a cts bdd fn.

<u>Ex.</u> $\mathbb{E} f(S_n) \longrightarrow \mathbb{E} f(N_\lambda)$ — $(*)$

$N_\lambda \stackrel{d}{=}$ Poisson(λ) r.v.

$(**)$ — $\mathbb{P}(S_n = k) \longrightarrow \mathbb{P}(N_\lambda = k)$ $\forall\, k \in \mathbb{N} \cup \{0\}$.

$(*) \iff S_n \xrightarrow{d} N_\lambda.$

convergence in dist
or weak convergence

$\Updownarrow$

$(**)$

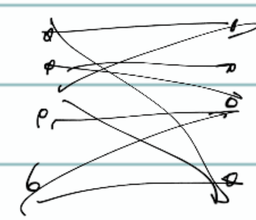
Identify a limit for S_n (N_λ in this case) & compute fn on the limit.

Back to 2 problems.

$$M_n^* = f((K_n, W_n)) \qquad W_n = (w_{i,j})_{1\le i<j\le n}.$$

$$C_n^* = g((K_{n,n}, C_{n,n})) \qquad C_{n,n} = (c_{ij})_{1\le i,j\le n}$$

$K_{n,n}$ - complete bipartite graph on $n \times n$ vertices



Qn: Can we find a limiting object to (K_n, W_n) or $(K_{n,n}, C_{n,n})$?

In case of S_n (real-valued) & notion of limits are well-defined via Euclidean metric.

But (K_n, W_n) or $(K_{2n}, C_{n,n})$ are weighted graphs.

What is a metric on this space?

<u>Local weak Convergence</u>:

Metric on "space of graphs" $\Rightarrow$ limit of a graph will be a graph.

If we consider "local functionals" of a graph these will be bdd & cts under the metric.

$$f(G) = \sum_{v \in G} \xi_v$$

Graph vertex

ξ_v ↳ depends only on "neighbourhood" of v

<u>Course</u> - compute $E f(G_n)$ ↳ when G_n is RG.

Aldous '99, 98 — Solved the 2 questions

$$(K_n, W_n) \longrightarrow (K^*, W^*)$$

$$\boxed{\mathbb{E} f(\quad) \longrightarrow \mathbb{E} f(\quad)}$$ converge

$$\zeta(2).$$