01/09/20 : LECTURE 1 :: Metric space of rooted graphs

$G = (V, E)$ - graphs. V - Vertex set, countable.

E - Edge set $\subseteq V \times V$

unordered pairs.

$v \sim w$ if $(v, w) \in E$.

neighbour, adjacent!

(Neighbourhood of v)

$N_G(v) = \{ w \in V : v \sim w \}$

$\deg_G(v) = \deg(v) = |N(v)|$

<u>locally finite</u> if $\deg_G(v) < \infty$, $\forall v \in V$.

<u>Rooted graph</u> : (G, o) is a rooted graph if G is a graph & $o \in V(G)$. o is called the root.

d_G - graph distance; $d_G(u, v) =$ length of shortest path between u & v $=$ (no. of edges)

$\rightarrow (G, d_G)$ is a metric space.

$H \subseteq G$ is a subgraph if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$.

$H = (V_1, E_1) \subseteq G$ is an induced subgraph if

$$E_1 = (V_1 \times V_1) \cap E(G)$$

i.e., all edges in G between vertices of H are present in H. Induced subgraphs are specified by vertex set alone.

$$B_r(v) = B_r^{(G)}(v) = \{w : d_G(v,w) \le r\}$$

We also use $B_r(v)$ to denote the induced subgraph on $B_r(v)$. i.e.,

$$E(B_r(v)) = \{(u,w) \in E : d_G(v,u) \le r, \; d_G(v,w) \le r\}$$

<u>Defn</u>: Graph isomorphism

$G_1 \cong G_2$ (G_1 is isomorphic to G_2) if $\exists$ a bijection $\phi : V_1 \to V_2 \ni$ $(i,j) \in E_1 \iff (\phi(i), \phi(j)) \in E_2$. ϕ is called graph isomorphism. ($\phi : G_1 \to G_2$, notation for simplicity)

$(G_1, o_1) \cong (G_2, o_2)$ if $\exists$ a graph isomorphism $\phi : G_1 \to G_2 \ni \phi(o_1) = o_2$.

$\phi : G_1 \to G_2$ is a graph homomorphism if $\phi : V_1 \to V_2$ & $(i,j) \in E_1 \Rightarrow (\phi(i), \phi(j)) \in E_2$.

$\mathcal{G}_*$ = space of rooted connected graphs modulo isomorphisms

= set of Equivalence classes of rooted connected graphs

$[(G,o)] \in \mathcal{G}_*$ but we'll use notation $(G,o) \in \mathcal{G}_*$ keeping in mind that $(G',o') \cong (G,o)$ are same.

<u>DEFN</u>: Let $(G_1,o_1), (G_2,o_2) \in \mathcal{G}_*$.

$$R^* = \sup\{r \geq 0 : B_r^{(G_1)}(o_1) \cong B_r^{(G_2)}(o_2)\}$$

$(R^* \geq 0)$

$$d_{\mathcal{G}_*}((G_1,o_1),(G_2,o_2)) = \frac{1}{R^*+1}$$

PROPERTIES OF $(\mathcal{G}_*, d_{\mathcal{G}_*})$: $d_{\mathcal{G}_*}$ is well-defined!

1. $d_{\mathcal{G}_*}$ is a metric space

(Lemma A.9 of VdH-2)

- $d_{\mathcal{G}_*} \geq 0$
- $d_{\mathcal{G}_*} = 0 \iff R_* = \infty \iff (G_1,o_1) \cong (G_2,o_2)$

Propn A.8 of vdH-2 : [ULTRAMETRICITY].

$$d_{\mathcal{G}_*}((G_1,o_1),(G_3,o_3)) \leq \max\{d_{\mathcal{G}_*}((G_1,o_1),(G_2,o_2)),$$
$$d_{\mathcal{G}_*}((G_2,o_2),(G_3,o_3))\}$$

$\Rightarrow (\mathcal{G}_*, d_{\mathcal{G}_*})$ is an ultrametric space

Sketch of proof: $R^*_{ij} = \sup\{r : B_r^{(G_i)}(o_i) \simeq B_r^{(G_j)}(o_j)\}$

$R^*_{12} \geq \min\{R^*_{13}, R^*_{23}\} \Rightarrow \cdots\cdots \blacksquare$

LEMMA: $(\mathcal{G}_*, d_{\mathcal{G}_*})$ is seperable.

Proof: $d(B_r^{G}(o), (G,o)) \leq \frac{1}{r+1} \to 0$ as $r \to \infty$.

Ex. $\Rightarrow$ seperability.

$S_* = \{$set of all ~~finite~~ graphs in $\mathcal{G}_*\}$
(finite V)

S_* is countable as $\exists$ finitely many eq. classes on graphs with n vertices, $\forall n$.

Given (G,o), $B_r^{(G)}(o) \in \mathcal{G}_*$

& $d\left((G,o), B_r^{(G)}(o)\right) \le \frac{1}{r+1}$. ∎

LEMMA A.11 of vdH-2.

Let $\{(G_r, o_r)\}_{r \ge 0}$ be connected loc. finite rooted graphs that are compatible i.e., $(B_r^{(G_s)}, o_s) \cong (B_r^{(G_r)}, o_r)$ $\forall\, r \le s$.

Then $\exists\,!$ (upto isomorphisms) (G,o)

$\ni \ (G_r, o_r) \simeq (B_r^{(G)}, o)$

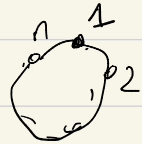
Propn A.10 of vdH-2 : $(\mathcal{G}_*, d_{\mathcal{G}_*})$ is a Polish space

A.6

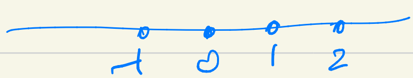
Examples of convergent sequences:

(1) $(G,o) \in \mathcal{G}_*$ $\quad B_r^{G}(o) \longrightarrow (G,o)$

(2) $(G_n, 1) = (C_n, 1)$ – cycle graph



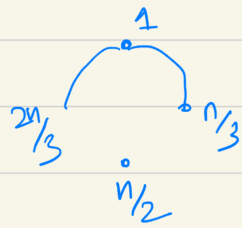
$(G_n, 1) \longrightarrow (\mathbb{Z}, 0)$.



$(\mathbb{Z}, -1) \cong (\mathbb{Z}, 1)$

$$R^* = \sup_q \{ r : B_r^{(G_n)}(1) \cong B_r^{(\mathbb{Z})}(0) \}$$

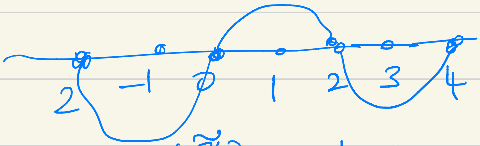
$$\geq \frac{n}{3}$$



$$\Rightarrow \quad d((G_n,1),(\mathbb{Z},0)) \to 0.$$

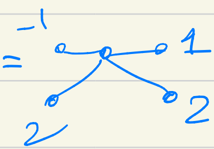
$(C_n,1)$ looks "locally like" $(\mathbb{Z},0)$.

3.) $(\mathbb{Z},0)$ $\quad (\tilde{\mathbb{Z}},0)$

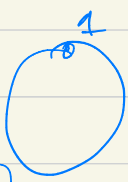


$$d((\mathbb{Z},0),(\tilde{\mathbb{Z}},0)) = 1$$

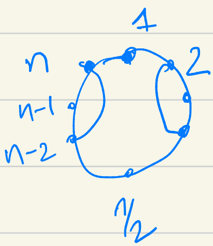
$B_1^{(\tilde{\mathbb{Z}})}(0) =$



4.)



$(C_n,1)$



$(\tilde{C}_n,1)$. $\quad$ n large.

$n-2 > \frac{3n}{4}$, $\quad 4 < \frac{n}{4}$.

$$d((C_n,1),(\tilde{C}_n,1)) = \frac{1}{2}.$$

$$(\tilde{C}_n,\tfrac{n}{2}) \to R^*((\tilde{C}_n,\tfrac{n}{2}),(C_n,1)) \geq \frac{n}{4}.$$

$$R^*((\tilde{C}_n,\tfrac{n}{2}),(\mathbb{Z},0)) \geq \frac{n}{4}.$$

$$(\tilde{C}_n, \tfrac{n}{2}) \longrightarrow (\mathbb{Z}, 0).$$

$$(\tilde{C}_n, v) \longrightarrow (\mathbb{Z}, 0) \quad \text{for "most" } v.$$

What we want to capture is for large n

$$(\tilde{C}_n, v) \approx (C_n, 1) \quad \text{for "most" } v.$$

Ex. (1) Fix $H_* \in \mathcal{G}_*$. $h: \mathcal{G}_* \longrightarrow \{0,1\}$

$$h(G,0) = \mathbb{1}[B_r(0) \cong H_*].$$

(2) $h: \mathcal{G}_* \longrightarrow \mathbb{N}$

$$(G,0) \longmapsto |B_r^G(0)|$$

$r=1 \Rightarrow |B_r^G(0)| = \deg(0) + 1.$

Are (1) & (2) cts?

what about $(G,0) \longmapsto f(B_r^G(0))$?

f = some fn on finite, rooted connected graphs.

takes values in some Polish space.

For eg: $(G,0) \mapsto (B_r^G(0), 0) \in \mathcal{G}_*$ i.e., $f = \mathrm{Id}$.

or $f : \mathcal{G}_* \to \mathbb{R}$ bdd fn. ?