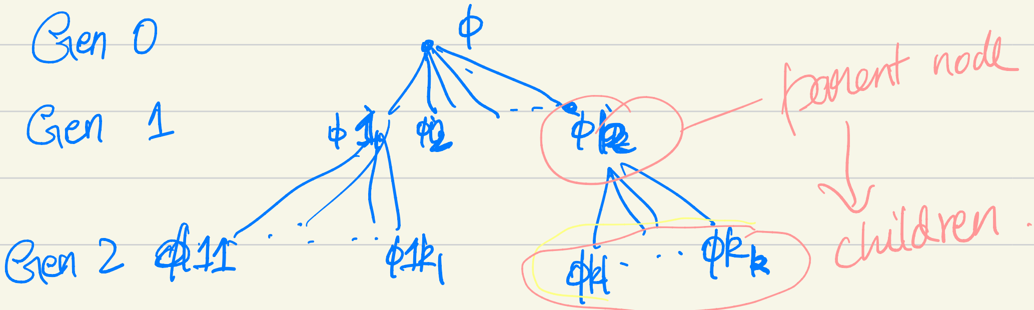
10/9:

L3 - Bienaymé-Galton-Watson Trees.

$\mathbb{N}^f$ — Set of possible tree nodes.

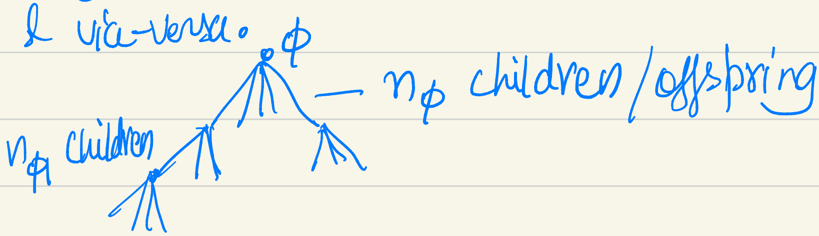
— $\{ (\phi i_1 \cdots i_k) : k \geq 0, \ i_j \geq 1 \}$

ϕ — root. $\quad v = (\phi i_1 \cdots i_k)$ is a vertex in generation k., $k = |v|$.



$\{ (\phi i_1 \cdots i_{k+1}) : i_{k+1} \geq 1 \}$ are called children of of the node $(\phi i_1 \cdots i_k)$

A sequence $\{ n_v \geq 0 : v \in \mathbb{N}^f \}$ specifies a rooted tree. & vice-versa.



BGW

Defn: BGW is a random tree in which each node has i.i.d. offspring distributed as N.
i.e., the offspring are specified by
$\{N_v : v \in \mathbb{N}^f\} \ni N_v$ are i.i.d. with the same distribution as N.

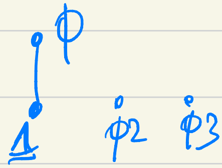
Notn: $p_k = P(N=k)$; $m = E[N] = \sum k p_k$

$$\phi_N(s) = \phi(s) = E[s^N] = \sum_{k=0}^{\infty} p_k s^k$$

$s \in [0,1]$.

$$BGW = \{ v \in \mathbb{N}^f : v = \phi i_1 \cdots i_k, \; i_1 \le N_\phi, \; i_2 \le N_{\phi i_1}, \; \ldots \ldots, \; i_k \le N_{\phi i_1 \cdots i_{k-1}} \}$$

Eg: $N_\phi = 1$, $N_{\phi 1} = 0$



$X_n = \#\{v \in BGW \mid |v| = n\}$ — no. of individuals in nth gen (vertices)

$X_1 = 1$, $X_2 = N_{(1)}$.

$$X_n = \sum_{\substack{v \in BGW \\ |v| = n-1}} N_v \overset{d}{=} \sum_{i=1}^{X_{n-1}} N_i$$

N_i i.i.d as N.

$S = \sum_{n=1}^{\infty} X_n$ — Total no. of individuals in BGW tree.

$X_n > 0 \iff X_n \geq 1$.

$X_n = 0 \implies X_{n+1} = 0$.

$\implies S < \infty \iff X_n = 0$ for some n

Extinction $= \{S < \infty\} = \{X_n = 0 \text{ for some } n\}$

$= \{BGW \text{ is a finite tree}\}$

$p_{ext} = \mathbb{P}(S < \infty) = \mathbb{P}(BGW \text{ is finite})$.

THM: P_{ext} is the smallest soln in s of the eqn $\phi(s) = s \quad \forall s \in [0,1]$

$$\phi(s) = \mathbb{E}[s^N]$$

Proof: $\{X_n = 0\} \uparrow$

$$\Rightarrow \quad P_{ext} = \mathbb{P}\Big(\bigcup_{n\geq 1} \{X_n = 0\}\Big) = \lim_{n\to\infty} \mathbb{P}(X_n = 0)$$

$$= \lim_{n\to\infty} \phi_{n+1}(0) = \lim_{n\to\infty} \phi_n(0)$$

$$\phi_{n+1}(s) = \mathbb{E}[s^{X_{n+1}}].$$

$$\phi_n(s) = \mathbb{E}[s^{X_n}] = \sum_{k=0}^{\infty} \mathbb{P}(X_{n-1} = k)\, \mathbb{E}\Big[s^{\sum_{i=1}^{k} N_i} \,\Big|\, X_{n-1} = k\Big]$$

(N_i's are indep of X_{n-1})

$$= \sum_{k=0}^{\infty} \mathbb{P}(X_{n-1} = k)\, \phi(s)^k$$

$$= \mathbb{E}\big[\phi(s)^{X_{n-1}}\big] = \phi_{n-1}(\phi(s))$$

induction

$$= \phi^n(s) = \phi(\phi_{n-1}(s))$$

$$P_{ext} = \lim_{n\to\infty} \phi_n(0) = \phi\left(\lim_{n\to\infty} \phi_{n-1}(0)\right) = \phi(P_{ext}).$$

[by cty of ϕ]

$\Rightarrow P_{ext}$ is a sol^n of $P_{ext} = \phi(P_{ext})$.

If $\eta \in [0,1]$ is another of $\eta = \phi(\eta)$.

$0 \le \eta$

ϕ is $\uparrow \Rightarrow \phi(0) \le \phi(\eta) = \eta$

$\Rightarrow \phi^n(0) \le \eta \Rightarrow P_{ext} \le \eta$. ∎

COR (1) If $m < 1$ then $P_{ext} = 1$ (Sub-critical)

(2) If $m > 1$ then $P_{ext} < 1$ (super-critical)

$\Rightarrow P(S = \infty) > 0$

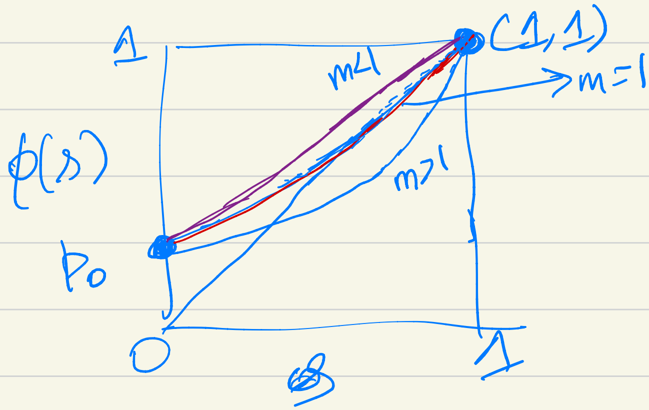
(3) $m = 1$ (critical)

$\rightarrow p_0 = 0$ ($\Leftrightarrow p_1 = 1$) then $P_{ext} = 0$

$\rightarrow p_0 > 0$ the $P_{ext} = 1$.

[$P_{ext} \ge p_0$. $p_0 = 0 \Rightarrow P_{ext} = 0$.]

Proof: $\phi(0) = p_0$, $\phi(1) = 1$.



$\Rightarrow$ ϕ is convex. $m = \phi'(1) > 0$.

$\Rightarrow$ ϕ has at most one soln < 1.

& if the sol^n exists then it is Pext.

$\Longleftrightarrow$ $m < 1$ or $m = 1$ & $p_0 > 0$.

[Cf. 2 ; B. Blaszczyszyn –] . ☒

$\mathbb{N}^f$ – has natural order. $u < v$ if $|u| < |v|$

$|u| = |v|$, then order by lexicographic ordering on indices.

For eg. $\phi 112 < \phi 113 < \phi 121$

Let $(D_i)_{i=1}^{n}$ be the no. of offsprings of the "first n" vertices in BGW.

$D_0 = N_\phi$.

LEMMA: $\mathbb{P}(D_i = d_i, 1 \le i \le n) = \prod_{i=1}^{n} P_{d_i}$

Notn: $BGW(\lambda)$ — BGW where $N \overset{d}{=} Poi(\lambda)$

$\rightarrow \mathbb{P}(D_i = d_i, 1 \le i \le n) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{d_i}}{d_i!}$ ∎