Sparse Random Graphs : Assignment 1

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1. Are the following functionals continuous ?
   (a) $h : \mathcal{G}_* \to \{0,1\}$ is defined as $h((G,o)) = 1[B_r^G(o) \cong H]$, where $r \geq 0$ and H is a connected graph.
   (b) $h : \mathcal{G}_* \to \{0,1\}$ is defined as $h((G,o)) = 1[|\partial B_1^G(o)| = l_1, \ldots, |\partial B_m^G(o)| = l_m]$ where $\partial B_r^G(o) = B_r^G(o) \setminus B_{r-1}^G(o)$ and $l_1, \ldots, l_m \in \{0,1,2,\ldots\}^m$.
   (c) $h : \mathcal{G}_* \to \{0,1\}$ is defined as $h((G,o)) = 1[|C(o)| \geq k]$ where $C(o)$ is the component of origin.

2. Let $D_i, i \in [n]$ be the degrees of the vertices in $G(n,p)$ for $p = \lambda/n, \lambda \in (0,\infty)$. Show that for any $m \geq 1$,

$$\lim_{n\to\infty} \mathbb{P}(D_i = k_i, 1 \leq i \leq m) = \prod_{i=1}^{m} \frac{e^{-\lambda}\lambda^{k_i}}{k_i!}.$$

3. Let H be a connected graph on k vertices , $k > 1$. For any subgraph $H_1 \subset H$, we denote $\frac{|E(H_1)|}{|V(H_1)|}$ by $d(H_1)$, called *the density*. Further, set $m(H) = \max\{d(H_1) : H_1 \subset H\}$. Show the following
   (a) If $p = o(n^{-1/m(H)})$ then $\mathbb{P}(H \subset G(n,p)) \to 0$.
   (b) If $p = \omega(n^{-1/m(H)})$ then $\mathbb{P}(H \subset G(n,p)) \to 1$.

4. Let T be a finite tree on k vertices. Let $X^*(T,G)$ denote the number of components in G isomorphic to T i.e.,

$$X^*(T,G) := \sum_{F \subset G; |V(F)|=k} 1[F \cong T]1[F \text{ is a component in } G].$$

   Let $G(n,p)$ be the ER random graph with $p = \lambda/n, \lambda \in (0,\infty)$. Show that $n^{-1}\mathbb{E}[X^*(T,G(n,p))]$ converges as $n \to \infty$ and also find the limit.

5. Show that for $p = \lambda/n, \lambda \in (0,\infty)$, we have that [1]

$$X(C_4, G(n,p)) \xrightarrow{d} Poi(\frac{\lambda^4}{8}).$$

[1] Anyone is welcome to try for general C_k.

6. Denote by $\phi_S(t)$ the probability generating function of the total number of nodes in the GW tree; $\phi_S(t) := \mathbb{E}[t^S]$ where S is the number of nodes in the GW tree. Show that

$$\phi_S(t) = t\phi_N(\phi_S(t)), s \in [0, 1],$$

where ϕ_N is the probability generating function of the off-spring random variable N.