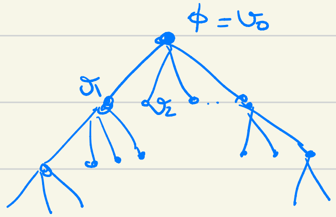
22/09/20 : L4 - Breadth-first exploration of Random graphs.

Exploration of BGW tree

$\mathcal{T}$ - tree



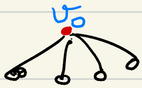
— Breadth-first ordering

$\mathcal{A}_k$ - Active nodes at step k ; ξ_k - no. of newly discovered nodes.

$A_k = |\mathcal{A}_k|$

$k=0$; $\mathcal{A}_0 = \phi$; $A_0 = 1$.

$k=1$; $\mathcal{A}_1 = N_{v_0} = \mathcal{A}_0 \cup N_{v_0} \setminus \{v_0\}$



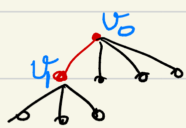
i.e, discover new nbhrs of v_0 & deactivate v_0.

$\xi_1 = |N_{v_0}|$ & $A_1 = A_0 + \xi_1 - 1 = \xi_1$.

$k=2$; Choose smallest node in $\mathcal{A}_1$ - say v_1.

Deactivate v_1 & discover all new nbhrs of v_1. children

$\xi_2 = |N_{v_1}| - 1$. $A_2 = A_1 + \xi_2 - 1$

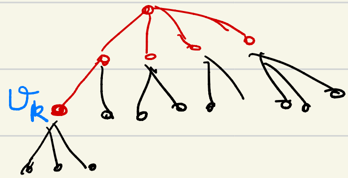


k – Choose smallest node in A_k – say v_k.

Deactivate v_k & discover all children of v_k.

ξ_k = # children of v_k = $|N_{v_k}| - 1$

$$A_k = A_{k-1} + \xi_k - 1.$$



Exploration can continue only as long as there are active nodes

i.e., $T = \min\{k \geq 1 : A_k = 0\}$. $\min \phi = \infty$.

$T < \infty$ iff $\mathcal{T}$ is finite.

Even more, $T = |\mathcal{T}|$, at each time step one node is de-activated.

Total de-activated nodes = Total discovered nodes

$$T = \sum_{k=1}^{T} \xi_k + 1.$$

[Easy for $T<\infty$ & holds for $T=\infty$ as well.]

Further if $\mathcal{T} \overset{d}{=} BGW(N)$ (offspring distribn) then ξ_k are iid. with distribn N. T & ξ_k's aren't independent!

So far the ordering of nodes do not matter!!!

Breadth-first is a convenient choice as it explores earlier generations / nbhrs of the root first.

i.e., if $d_\tau(v,\phi) < d_\tau(w,\phi)$ then v becomes active before w.

- $\mathcal{H} = (\xi_1, \dots, \xi_T)$ — History of the tree $\mathcal{T}$.

$\exists$ Bijn between $\mathcal{H}$ & $\mathcal{T}$.

Also, $(x_1, \dots, x_k)$ with $x_i \geq 0$, $1 \leq k \leq \infty$ is a (valid) realization of history $\mathcal{H}$ iff

$$\sum_{i=1}^{l} x_i + 1 > l \quad \forall\, l = 1, \dots, k-1$$

(if $\exists$ active nodes, more discovered than deactivated)

$$\sum_{i=1}^{k} x_i + 1 = k.$$

(if $\exists$ no active nodes discovered = deactivated).

$$\mathbb{P}(\mathcal{H} = (x_1, \dots, x_k)) = \prod_{i=1}^{k} p_{x_i} \quad \text{for BGW}(N)$$

$$p_k = P(N=k).$$

→ Very useful tool — Encodes a tree as a sequence of random variables.

Or as a random walk with step distribution $\xi_i - 1$.

Easy

<u>Appns:</u> Super-critical BGW | {extinction} = Sub-critical BGW.

Thm 2.2.5 of Blaszczyszyn.

EXPLORATION OF A GRAPH. Fix an ordering.

Let $v_0 = v$.

$k=0$. $A_0 = \{v\}$, $\mathcal{B}_0 = \phi$ (discovered but inactive i.e., deactivated)

$k=1$, Pick $v_1 = v$. deactivate it.

Set $\mathcal{D}_1 = \mathcal{D}(v_1) = N(v_1)$

$A_1 = A_0 \cup \mathcal{D}_1 \setminus \{v_1\}$.

$\mathcal{B}_1 = \{v_1\}$.

$\xi_1 = |\mathcal{D}_1|$ - Label $\mathcal{D}_1$ as $v_2, \dots, v_{\xi_1}$ as per initial order.

Step k: If $A_{k-1} \neq \phi$, pick v_k. Deactivate v_k.

Set $\mathcal{D}_k = \mathcal{D}(v_k) = N(v_k) \setminus (A_{k-1} \cup \mathcal{B}_{k-1})$.

$A_k = A_{k-1} \cup \mathcal{D}_k \setminus \{v_k\}$

$\mathcal{B}_k = \mathcal{B}_{k-1} \cup \{v_k\}$.

$A_k = |A_k|$, $B_k = |\mathcal{B}_k|$, $\xi_k = |\mathcal{D}_k|$

$\Rightarrow$ $A_0 = 1$; $B_0 = 0$ . . . $B_k = k$.

$A_k = A_{k-1} + \xi_k - 1$.

Suppose G is a graph on n vertices then knowing $\xi_1, \dots, \xi_k$ restricts choices for ξ_{k+1}

So if $G = G(n,p)$, $\xi_1, \ldots, \xi_k$ aren't independent.

Also $\xi_i \neq |N(v_i)|$!

Recall from assignment 1, $\forall\, m \geq 1$

$$\mathbb{P}(D_1 = k_1, \ldots, D_m = k_m) \to \prod_{i=1}^{m} e^{-\lambda} \frac{\lambda^{k_i}}{k_i!} \quad \left(p = \frac{\lambda}{n}\right)$$

$D_i = \deg(i)$

i.e., if we pick m arbitrary vertices & look at their degrees they are indep. $\mathrm{Poi}(\lambda)$ r.v.'s!

But in a BGW(*), even if we look at the first 'm' nodes this is true!

We'll show this for $G(n,p)$, $p = \frac{\lambda}{n}$ now.