29/09 L5 - Weak convergence & LWC.

Weak Convergence: [Ref: Sec 3.1 of CB].

DEF: $X, X_n, n \geq 1$ sequence of random variables

$X_n \xrightarrow{d} X$ (convergence in distribution or weak convergence)

if $F_{X_n}(x) \longrightarrow F_X(x)$ $\forall x$ where F_X is cts.

($F_X(x) = \mathbb{P}(X \leq x)$, CDF of X)

$\rightarrow$ $X_n \xrightarrow{d} X$ iff $\mathbb{E} f(X_n) \longrightarrow \mathbb{E} f(X)$ $\forall$ bdd cts fns.

This motivates a general definition of weak convergence of random elements in a metric space or prob. measures on a metric space.

Recall $\mathbb{E} f(X) = \int f(x) \mathbb{P}_X(dx)$, $\mathbb{P}_X(A) = \mathbb{P}(X \in A)$ prob. distribn of X.

DEF: Let X_n, X be random elements in a Polish space $(S, \mathcal{S})$. Then $X_n \xrightarrow{d} X$ or $\mathbb{P}_{X_n} \xrightarrow{d} \mathbb{P}_X$ if

$$\mathbb{E} f(X_n) \longrightarrow \mathbb{E} f(X) \quad \forall \text{ bdd cts } f.$$

THM (Portmanteau Theorem)

TFAE

(1) $X_n \xrightarrow{d} X$.

(2) $\mathbb{E} f(X_n) \to \mathbb{E} f(X)$ $\forall$ bdd, unif. cts fns f.

(3) $\overline{\lim}\, \mathbb{P}(X_n \in F) \leq \mathbb{P}(X \in F)$ $\forall$ closed sets F

(4) $\underline{\lim}\, \mathbb{P}(X_n \in G) \geq \mathbb{P}(X \in G)$ $\forall$ open sets G

(5) $\lim \mathbb{P}(X_n \in A) = \mathbb{P}(X \in A)$ $\forall A \in \mathcal{S}$ $\ni$ $\mathbb{P}(X \in \partial A) = 0$.

Rmk: $\underset{\substack{\shortparallel \\ \mathbb{P}(X \in (-\infty, x])}}{F_X(x)}$ is cts at x iff $\mathbb{P}(X = x) = 0$.

$X_n \xrightarrow{d} X$ = Convergence in distribution

or

weak convergence.

$\mathcal{G}_*$ — space of loc-finite rooted (connected) graphs.

G — graph disconnected & C_0 — conn. component of 0

$$(G, 0) = (C_0, 0)$$

$$C_0 = \quad V(C_0) = \{u : d_G(0, u) < \infty\}$$

$$E(C_0) = \{(u,v) \in E(G) : u, v \in V(C_0)\}$$

DEFN (LWC): Let G_n be a finite (can be disconn.) graph.

Let $0_n \overset{d}{=} \mathrm{Unif}(V_n)$ with $V_n = V(G_n)$, $E_n = E(G_n)$.

We say $G_n \xrightarrow{LW} (G, 0)$ ($(G, 0)$ is a random element of $\mathcal{G}_*$ with prob. dist. μ)

[G_n converges to $(G, 0)$ in local weak topology]

if $$\mathbb{E}_n[h(G_n, 0_n)] \to \mathbb{E}_\mu[h(G, 0)]$$

$\forall$ bdd cts $h : \mathcal{G}_* \to \mathbb{R}$.

$$\mathbb{E}_n\left[h(G_n, o_n)\right] = \frac{1}{n} \sum_{u \in [n]} h(G_n, u)$$

$\boxed{\text{Remarks:}}$ (1) Convergence of a graph to a rooted graph

(2) By defn, it means from most vertices, the graph "looks like" (G, o) "locally" for large n.

(3) Even if we consider G_n to be deterministic graphs, it is possible that (G, o) is random.

<u>THM:</u> Let G_n be as above. $G_n \xrightarrow{LW} (G, o)$ iff $\forall\, H_* \in \mathcal{G}_*$ we have that

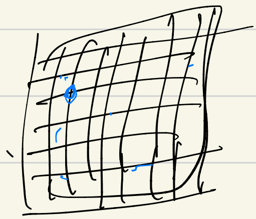
$$\frac{1}{n} \sum_{u \in [n]} \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_*\right] \to \mathbb{P}\left(B_r^{G}(o) \cong H_*\right)$$

$G_n \xrightarrow{LW} G \implies \forall H_* \in \mathcal{G}_*$

$$\frac{1}{n}\sum_{u\in[n]} \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_*\right] \to \mathbb{P}\left(B_r^{(G)}(o) \cong H\right)$$

[Take $h(G,u) = \mathbb{1}\left[B_r^{(G)}(u) \cong H\right]$
Add details of Assignment 1]

<u>Eg:</u> $G_n = \mathbb{Z}^d \cap [-n,n]^d$



$G_n \xrightarrow{LW} ?$

$(G_n, 0) \longrightarrow (\mathbb{Z}^d, 0) \quad n \to \infty.$

<u>First:</u> $W_n = [-n,n]^d$ $\quad \underline{W_{n,r}} = \{v \in V_n : d(v, \partial W_n) \le r\}$

$$\frac{|V_n \cap W_{n,r}|}{|V_n|} \longrightarrow 1 \quad \text{as } n \to \infty.$$

'cause $|V_n| \sim (2n)^d$ $\quad |W_n \setminus W_{n,r}| \le C_d\, n^{d-1}\, r$

$$\Rightarrow \quad \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_*\right] = \mathbb{1}\left[B_r^{\mathbb{Z}^d}(0) \cong H_*\right]$$

$$\forall u \in W_{n,r}$$

$$\Rightarrow$$

$$\frac{1}{|V_n|} \sum_{u \in V_n} \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_*\right]$$

$$= \frac{\mathbb{1}\left[B_r^{\mathbb{Z}^d}(0) \cong H_*\right] |V_n \cap W_{n,r}|}{|V_n|}$$

$$+ \underbrace{\frac{1}{|V_n|} \sum_{u \notin W_{n,r}} \mathbb{1}[\cdots]}_{\to 0}.$$

$$\longrightarrow \mathbb{1}\left[B_r^{(\mathbb{Z}^d)}(0) \cong H_*\right].$$

$$\Rightarrow \quad G_n \xrightarrow{LW} (\mathbb{Z}^d, 0).$$

<u>Ex (A)</u> Amenable Cayley Graphs.

<u>Ex.</u> $\mathbb{T}_n^d = \left[\mathbb{Z}^d \cap [0, n]\right]^d / \sim$ (discrete torus)

$$= (\mathbb{Z}/n\mathbb{Z})^d$$

$$(\mathbb{T}_n^d, u) \cong (\mathbb{T}_n^d, v)$$

$$\Rightarrow \quad \mathbb{T}_n^d \xrightarrow{LW} (\mathbb{Z}^d, 0) .$$