Due: Thursday, October 1st

Let G be a graph with n vertices.

1. Let F be a fixed graph with k vertices.
   (a) Suppose K_n is the complete graph on n vertices then show that
   $$\mathrm{emb}(F, K_n) = \prod_{i=0}^{k-1} (n - i),$$
   where $\mathrm{emb}(F, K_n)$ is the number of injective homomorphisms of F into G.
   (b) Show that
   $$\mathrm{emb}(F, G) = \mathrm{aut}(F) X_F(G),$$
   where $\mathrm{aut}(F, G)$ is the number of automorphisms of F into F and $X_F(G)$ is the number of subgraphs of G isomorphic to F.
2. Suppose $\hom(F, G)$ are the number of homomorphisms from F to G then show that
   $$t(F, G) := \frac{\hom(F, G)}{n^k} = s(F, G) + O(\frac{1}{n}),$$
   with $s(F, G) = \frac{X_F(G)}{X_F(K_n)}$
3. Let $\mathcal{F}$ denote the set of isomorphism classes of finite graphs eumerated by $\{F_1, F_2, \ldots\}$, with each F_i being a representative of an isomorphism class. Define:
   $$d_{\mathrm{sub}}(G, G') = \sum_{i \geq 1} 2^{-i} \big| s(F_i, G) - s(F_i, G') \big|.$$
   (a) Show that $(d_{\mathrm{sub}}(G, G'), \mathcal{F})$ is a discrete metric space.
   (b) If we replace s with t in the above definition of d_{sub} then do we obtain a metric on $\mathcal{F}$?
4. Let $\kappa : [0, 1]^2 \to [0, 1]$ be symmetric and measurable. Extend s to κ by:
   $$s(F, \kappa) = \int_{[0,1]^k} \prod_{\{i,j\} \in E(F)} \kappa(x_i, x_j)\, dx_1 \cdots dx_k,$$
   Divide $[0, 1]$ into n intervals $I_1, \ldots I_n$ of equal length (ignore end points), and set $\kappa_G : [0,]1^2 \to [0, 1]$ to be given by
   $$\kappa_G(x, y) = \begin{cases} 1 & \text{if } (x, y) \in I_i \times I_j \text{ and } (i, j) \text{ is an edge in } G. \\ 0 & \text{otherwise.} \end{cases}$$
   Show that $t(F, G) = s(F, \kappa_G)$.