Dense graphs :-

$V = \{1, \dots, n\}$

$E \subseteq V \times V$ $\quad E = \{(i,j) \mid i \in V, j \in V\}$

$G = (V, E)$

G is called dense if $\# E = \Theta(n^2)$

The Sub-graph distance :-

F - fixed graph

# of copies of F in a large graph G

i.e. the number, $X_F(G)$ of subgraphs of G isomorphic to F.

[isomorphism $f : V(F) \to V(G)$

u, v are adjacent in F $\iff$ $f(u), f(v)$ are adjacent in G]

A $\phi : V(F) \to V(G)$ is a homomorphism if $(\phi(x), \phi(y))$ are edges in G whenever (x, y) are edges in F

$\text{emb}(F, G) :=$ the number of injective homomorphism of F into G

Ex :- $\quad \text{emb}(F, G) = \text{aut}(F)\, X_F(G)$

[automorphism : $\varphi : F \to F$ φ - onto and preserves edges and vertices]

So $X_F(G)$ and $\mathrm{emb}(F,G)$ contain the same information.

- F has k - vertices and $n \geq k$.

[K_n :- Complete graph on n- vertices]

Ex :- $\mathrm{emb}(F, K_n) = n(n-1)\cdots(n-k+1) := n_{(k)}$

Natural normalisation :- $G = (V, E)$ $|V| = n$

$$s(F, G) = \frac{\mathrm{emb}(F,G)}{\mathrm{emb}(F, K_n)} = \frac{\mathrm{emb}(F,G)}{n_{(k)}} \quad (*)$$

if $|F| \leq |G|$.

$s(F, G) = 0$ otherwise.

In $(*)$ $$s(F, G) = \frac{X_F(G)}{X_F(K_n)} \in [0,1]$$

[I may have used $t(F, G) :=$ caution]

let $\mathcal{F}$ = set of isomorphism classes of finite graph.

$\mathcal{F} = \{F_1, F_2, \ldots\}$ where each $F_i \equiv$ representative of an isomorphism class.

$S(F, \cdot)$, $F \in \mathcal{F}$:= natural family of equivalent metrics on $\mathcal{F}$, by mapping

$$\mathcal{F} \longrightarrow [0,1]^{\infty} \quad (\equiv [0,1]^{\mathcal{F}})$$

Take any graph G (finite)

$$S(G) = (S_i(G))_{i=1}^{\infty} \in [0,1]^{\infty}$$

where $S_i(G) = S(F_i, G)$.

Take any metric d that provides product topology: (e.g.) $d(s,t) = \sum_{i=1}^{\infty} \frac{1}{2^i} |s_i - t_i|$ for $t, s \in [0,1]^{\infty}$

$$d_{sub}(G_1, G_2) = d(S(G_1), S(G_2)).$$

Ex: d_{sub} – a metric on $\mathcal{F}$.

- Further, $(\mathcal{F}, d_{sub})$ is a discrete metric space

[Sketch :- Given $G \in \mathcal{F}$ among graph F

- with $S(F,G) > 0$ there is a unique graph with $|F| + e(F)$ maximal, namely G.

- $E_n \equiv$ empty graph with n vertices

$S(E_n, G) > 0$ if G has n vertices or more]

- A sequence $(G_n)_{n\geq 1}$ of graphs is Cauchy w.r.t. $d_{sub}(\cdot,\cdot)$ iff for each F

$$\{S(F, G_n)\}_{n\geq 1} \text{ — converges.}$$

$\therefore$ As $(\mathcal{F}, d_{sub})$ — discrete, if $(G_n)_{n\geq 1}$ is a Cauchy sequence then either (G_n) is eventually constant or $|G_n| \longrightarrow \infty$ $\quad (|G| \equiv \#$ of vertices$)$

Recall :-

$|F| = k$, F - fixed graph G - given graph

$$s(F,G) = \frac{X_F(G)}{X_F(K_n)} = \frac{emb(F,G)}{n_{(k)}}$$

where $n_{(k)} = n(n-1)\cdots(n-k+1)$ if $|F| < |G|$

$$t(F,G) = \frac{hom(F,G)}{n^k}$$

($\because$ # of non-injective homomorphism F to G $< \binom{k}{2} n^{k-1}$)

$$t(F,G) = s(F,G) + O\left(\frac{1}{n}\right).$$

· Graphon $k : [0,1]^2 \to [0,1]$

$$s(F,k) = \int_{[0,1]^k} \prod_{i,j \in e(F)} k(x_i, x_j) \prod_{i=1}^{k} dx_i$$

· $d_{sub}(G_1, G_2) = \sum_{i=1}^{\infty} \frac{1}{2^i} |s(F_i, G_1) - s(F_i, G_2)|$

$(F_i)_{i \geq 1}$ of isomorphism classes - $\mathcal{F}$.

· $(\mathcal{F} \cup K - , d_{sub})$ - Complete / Compact metric space.

K_1, K_2 are two graphons.

Q: when are $S(F, K_1) \neq S(F, K_2)$ $\forall F \in \mathcal{F}$?

or Define $\sim$ equivalence relation $K_1 \sim K_2 \iff$

$S(F,K_1) = S(F,K_2)$ $\forall F \in \mathcal{F}$.

The cut distance :-

Another natural metric on graphs or graphons or kernels

$K : [0,1]^2 \to \mathbb{R}$ — its cut norm

$$\|K\|_{cut} = \sup_{S,T \subseteq [0,1]} \left| \int_{S \times T} K(x,y)\,dx\,dy \right|$$

over all measurable subsets of $[0,1]$

Ex:- Is indeed a norm on Graphons ; $L^\infty([0,1]^2)$

Variations Exist :-

- $$\|K\|_{cut} = \sup_{S \subseteq [0,1]} \left| \int_{S \times S^c} K(x,y)\,dx\,dy \right|$$

- $$\|K\|_{cut} = \sup_{\|f\|_\infty, \|g\|_\infty \le 1} \int_{[0,1]^2} K(x,y) f(x) g(y)\,dx\,dy$$

$f, g : [0,1] \to [-1,1]$ - measurable

$\boxed{\text{Metric}}$ on Graphons: $k_1, k_2 : [0,1]^2 \to [0,1]$

$$d_{cut}(k_1, k_2) = \| k_1 - k_2 \|_{cut}$$

- Given a $k : [0,1]^2 \to [0,1]$ $\tau : [0,1] \to [0,1]$ - a measure preserving map.

$$k^{\tau}(x,y) = k(\tau(x), \tau(y))$$

- $k_1 \approx k_2$ if $\exists$ a measuring preserving $\tau : [0,1] \to [0,1]$ st

$$k_1(x,y) = k_2^{\tau}(x,y) \quad \text{for a.e } (x,y) \in [0,1]^2$$

Ex: Equivalence relation.

If $k_1 \approx k_2 \implies d_{cut}(k_1, k_2) = 0$.

(Reverse implication is not true)

- Metric on graphs :-

$$d_{cut}(G_1, G_2) := d_{cut}(k_{G_1}, k_{G_2})$$

Idea: If $\sim$ relation is identified then could have. $\left(\mathcal{F} \cup k, \ d_{cut} \right)$ - metric space.

Lemma: let k_1, h_1 be two graphons. Then for every graph F we have, $|F| = k$ - vertices

$$\| S(F, k_1) - S(F, h_1) \| \leq \# e(F) \, \| k_1 - h_1 \|_{cut}$$

Proof:

$$S(F, k_1) = \int_{[0,1]^k} \prod_{(i,j) \in e(F)} k_1(x_i, x_j) \prod_{i=1}^{k} dx_i$$

Say m edges of $F = \{ (i_1, j_1), \ldots, (i_m, j_m) \}$

$$S(F, k_1) = \int_{[0,1]^k} \prod_{r=1}^{m} k(x_{i_r}, x_{j_r}) \prod_{i=1}^{k} dx_i$$

Extend: $S(F; k_1, \ldots k_m)$ where $k_e : [0,1]^2 \to [0,1]$ symmetric m'ble

$$S(F; k_1, \ldots k_m) = \int_{[0,1]^k} \prod_{r=1}^{m} k_r(x_{i_r}, x_{j_r}) \prod_{i=1}^{k} dx_i$$

Note:- $S(F; \underbrace{k, \ldots, k}_{m}) = S(F, k)$

Claim:- Give $k_1, k_2 \ldots k_m$ & h_1 graphons:

$$| S(F; k_1, k_2, \ldots, k_m) - S(F; h_1, k_2, \ldots, k_m) | \leq \| k_1 - h_1 \|_{cut} \quad - (*)$$

Given claim:-

$$S(F, k_1) = S(F; k_1, \ldots k_1)$$

$$S(F, h_1) = S(F, h_1 \ldots h_1)$$

$$|S(F, k_1) - S(F, h_1)| \leq \ldots$$ (∗) – m times replace $m = \#e(F)$ times, one change

and use the claim to get

$$|S(F, k_1) - S(F, h_1)| \leq \#e(F) \, \|k_1 - h_1\|_{cut}$$

Proof of claim :-

WLOG the first edge in F $= (1,2)$. $i_1 = 1$, $j_1 = 2$

$$\Delta = \int_{[0,1]^k} \left[k_1(x_1, x_2) - h_1(x_1, x_2)\right] \prod_{l=2}^{m} k_l(x_{i_l}, x_{j_l}) \prod_{i=1}^{k} dx_i$$

$$\mathbf{X} = (x_3, \ldots x_k)$$

product in integrand :- $f_0(\mathbf{X}) \, f_1(x, \mathbf{X}) \, f_2(x_2, \mathbf{X})$

where f_0, f_1, f_2 are products of graphons

range $f_0, f_1, f_2 \subseteq [0,1]$.

Recall :-

- $$\|k\|_{cut} = \sup_{\|f\|_\infty \, \|g\|_\infty \leq 1} \int_{[0,1]^2} k(x,y) f(x) g(y) \, dx \, dy$$

$f, g : [0,1] \to [-1,1]$ – measurable

$$\Rightarrow |\Delta| \leq \|k_1 - h_1\|_{cut} \Rightarrow \text{claim } \square.$$

Major break through :- Lovasz & Szegedy ++

- $d_{sob}(h_n, k) \to 0 \iff d_{cut}(h_n, k) \to 0$

$\Rightarrow$ If k_1 & h_1 are two graphons then

- $S(F, k_1) = S(F, h_1) \iff d_{cut}(k, h_1) = 0$ — (+)

Equivalence Relation :- $\sim$:

$k_1 \sim k_2$ iff $\exists$ $\sigma_1 : [0,1) \to [0,1)$
$\sigma_2 : [0,1) \to [0,1)$

measure preserving maps such that

$\exists$ $k : [0,1)^2 \to [0,1)$ - graphon

with $k_1^{\sigma_1}(x, y) = k(x, y)$ are (x,y)

& $k_2^{\sigma_2}(x, y) = k(x, y)$ a.e. $\in [0,1)^2$

Show: $k_1 \sim k_2 \iff d_{cut}(k_1, k_2) = 0$ — $(*)$

$$\mathcal{K} = \{ [k] \mid k : [0,1]^2 \to [0,1] \text{ graphon} \}$$

$[k]$ - equivalence $\sim$

$((\mathcal{F} \cup \mathcal{K}), d_{sub})$ - metric space. — $(*)$ & (f).

Cor :- $(G_n)_{n \geq 1}$ be a sequence of graphs with $|G_n| \to \infty$ and k - graphon.

If $d_{cut}(G_n, k) \to 0$ then

$$d_{sub}(G_n, k) \to 0.$$

Proof :- let F be any fixed graph.

Use lemma :- $|S(F, k_1) - S(F, h_1)| \leq \#e(F) \| h_1 - k_1 \|_{cut}$

Given :- $\| K_{G_n} - k \|_{cut} \to 0$ as $n \to \infty$.

By lemma :- $S(F, K_{G_n}) \longrightarrow S(F, k)$ — $(*)$

as $n \to \infty$.

But : $S(F, K_{G_n}) = t(F, G_n)$ — (Ex last class)

(Ex.) $S(F, G_n) = t(F, G_n) + O\left(\frac{1}{n}\right)$ — $(**)$

(*) and (**) $\Rightarrow$ $S(F, G_n) \longrightarrow S(F, K)$
as $n \to \infty$.

$\Rightarrow$ $d_{sub}(G_n, K) \longrightarrow 0$ as $n \to \infty$ □

Random graphs from graphons :-

$K : [0,1]^2 \longrightarrow [0,1]$

$V_n = \{1, \ldots n\}$, E_n: $i \sim j$ w.p. $K(U_i, U_j)$

$U_1 \quad U_n$ – uniform $[0,1]$
i.i.d labels

$G(n, K) = (V_n, E_n)$ [Random graph]

Show :- $d_{sub}(G(n,K), K) \longrightarrow 0$
as $n \to \infty$ w.p. 1.