# (Random) Finite graphs from Graphons :-

$K : [0,1]^2 \longrightarrow [0,1]$ - Symmetric and measurable

n-vertices $\{1, \dots, n\} \equiv V$

Edge $i \sim j$ with probability $K(U_i, U_j)$ $\equiv E$

when $\{U_i\}_{i=1}^{n}$ are independent $U[0,1]$ random variables.

$$\mathcal{G}(n, K) \equiv (V, E).$$

$\boxed{i \sim j \quad \text{with probability}}$ $K(U_i, U_j)$

$V = \{1, \dots, n\}$

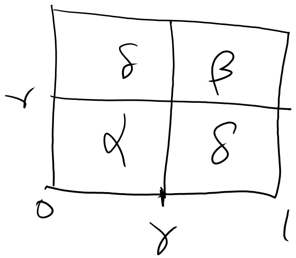
$\boxed{\text{labels} \quad u_1 \quad \dots \quad u_n}$ $\equiv U(0,1)$

Examples :- • $k(\cdot,\cdot) = p$ $\quad 0<p<1$

Then $G(n,p) \equiv$ Erdős – Rényi

– Each edge $i \sim j$ w.p. $p$

$V = \{1, \dots, n\}$

• 


$G(n,k)$ $\quad V = \{1 \;\dots\; n\}$

$\uparrow \quad\quad \uparrow$

$u_1 \quad \dots \quad u_n$

$i \sim j$ with probability $k(u_i, u_j)$

$A = \{i \in V \mid u_i < r\}$. $\quad B = \{i \in V \mid u_i > r\}$.

Show :- $G(n,k) \overset{?}{\dashrightarrow} K$ in $d_{sub}$ as $n \to \infty$

Theorem : Let $\{U_i\}_{i\geq 1}$ be a sequence of i.i.d uniform $[0,1]$ random variables. Let $K : [0,1]^2 \to [0,1]$ be a graphon and be continuous a.e on $([0,1] \times [0,1], dx \times dy)$

Then $d_{sub}(G(n,k), K) \xrightarrow[\text{as } n\to\infty]{} 0$ with probability 1.

Remark :- • $U^{(n)} = (U_1^{(n)}, \ldots, U_i^{(n)})$ where $\{U_i^{(j)}\}_{\substack{j\geq 1\\ i\geq 1}}$ are i.i.d $U(0,1)$ – Theorem ✓

(Resample edges at each $n$)

• ~~Proof that we will~~ (? Result) presnt breaks down if Continuity assumption does not hold.

• [LS 06] + Theorem $\Rightarrow$ There is always an "intertwined" random graph sequence that converges to the graphon for any Cauchy sequence $\{G_n\}_{n\geq 1}$ of dense graphs.

Proof :- To show $d_{sub}(G(n,k),k) \longrightarrow 0$ as $n \to \infty$ w.p. 1

where:

$$d_{sub}(G(n,k),k) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left| S(F_i, G(n,k)) - S(F_i,k) \right|$$

# of vertices $F_i$ $\equiv m$

$$S(F_i, G(n,k)) = \frac{X_{F_i}(G(n,k))}{X_{F_i}(K_n)}$$

$$S(F_i,k) = \int_{[0,1]^m} \prod_{j=1}^{m} dx_j \prod_{(l,u)\in e(F)} k(x_l, x_u)$$

Its enough to show:

$S(F_i, G(n,k)) \longrightarrow S(F_i,k)$ as $n\to\infty$

with probability one.

$\mathbb{E}$ - expectation over the edge probabilities

$E$ - expectation over the labels.

Condition on $u_1, u_2, \ldots u_n = \underbrace{(x_1, \ldots, x_n)}_{:= x} \in [0,1]^n$

let $G(n,k)\Big|_{(u_1,\ldots u_n) = (x_1,\ldots x_n)} :\equiv G(n, x, k)$

let $F$ be a finite graph.

Q: How large is $|S(F, G(n,x,k)) - \mathbb{E}(S(F, G(n,x,k)))|$ as $n \to \infty$?

$\left.\begin{array}{l}\text{Concentration - inequalities} \\ \text{can be used}\end{array}\right\}$ to understand the above

## McDiarmid's Concentration Inequality

$X_1, X_2, \dots X_n$ be independent, $X_i \in \mathcal{X}_i$

"Lipschitz":

$f : \prod_{i=1}^{n} \mathcal{X}_i \longrightarrow \mathbb{R}$ that is

i.e $x, x' \in \prod_{i=1}^{n} \mathcal{X}_i$ that differ only in the $k^{th}$ coordinate then

$$|f(x) - f(x')| \le \sigma_k.$$

let $Y = f(X_1, \dots, X_n)$. Then $\forall \alpha > 0$

$$\mathbb{P}(|Y - E[Y]| > \alpha) \le 2 \exp\left(-\frac{2\alpha^2}{\sum_{i=1}^{k} \sigma_i^2}\right)$$

$S(F_i, G(n,x,k))$ = function on all the edges.

- random variables in the Theorem are Edge random variables

Q - How much will $S(F_i, G(n,x,k))$ change if we remove / add a single edge?

- Bound for change

ANS :- F - finite graph - on k vertices

G - graph on n-vertices

G' - graph on n-vertices

G' differs from G by a single edge

Ex :-

$$|S(F,G) - S(F,G')| \leq \frac{k(k-1)}{n(n-1)} \equiv \sigma$$

same $\forall$ edge change

Fix $\varepsilon > 0$ :- $\mathbb{P}\left(\left|S(F, G(n,x,k)) - \mathbb{E}[S(F, G(n,x,k))]\right| > \varepsilon\right)$

F - graph on k - vertices

$$\leq 2\exp\left[-\frac{2\varepsilon^2}{{}^{n}C_2\left(\frac{k(k-1)}{n(n-1)}\right)^2}\right]$$

$$= 2\exp(-a_n) \qquad a_n = O(n^2)$$

$$\implies$$

Ex :- $\left|S(F, G(n,x,k)) - \mathbb{E}[S(F, G(n,x,k))]\right|$

$\longrightarrow 0$ with probability one

To show the result we need to show:

$$S(F, G(n,k)) \equiv S(F, G(n,(u_1 \ldots u_n), k))$$

We showed:

$$\left| S(F, G(n, u_1, \ldots u_n, k)) - E(S(F, G(n, u_1, \ldots u_n, k)))\right| \longrightarrow 0$$

with probability one

We need to show:

Generalised U-statistic

$$E(S(F, G(n, u_1, \ldots u_n, k))) \xrightarrow{LLN} S(F, k)$$

with probability 1

$$S(F, k) = E\Big(\prod_{(i,j) \in e(F)} k(v_i, v_j)\Big)$$

independent

$$V_i = U(0,1)$$

i.i.d.'s

"U-statistic"

Please:

Zoom Contacts

↓ add

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as contact