Variations to definition of d_{sub}

Instead of embeddings – one can consider number of homomorphism

F – fixed graph. For a given graph G, $\hom(F,G) :=$ # of homomorphism between F and G.

If $|F| = k$, $|G| = n$ the number of non-injective homomorphism from F to G is at most

$$\binom{k}{2} n^{k-1} = O(n^{k-1})$$

Set

$$t(F,G) = \frac{\hom(F,G)}{n^k}$$

Ex:- $t(F,G) = s(F,G) + \frac{c_k}{n}$

In dense graph setting t and s equivalent notions. If in d_{sub} we replace s with t then we obtain a pseudo metric. [G – $G^{(r)}$ = blow up of G by making r copies of each vertex joined to all copies of its neighbours $t(F,G) = t(F, G^{(r)})$ $\forall F \in \mathcal{F}$] $r \geq 1$

Completing $(\mathcal{F}, d_{sub})$:-

d_{sub} — provides a natural way to complete $(\mathcal{F}, d_{sub})$. This is done with the notion of graphons.

A $K : [0,1]^2 \to [0,1]$ is called a graphon if

K is symmetric $K(x,y) = K(y,x)$

and measurable

[Standard kernels : $K : [0,1]^2 \to [0, \infty)$
Signed kernels $K : [0,1]^2 \to [-M, M]$]

Extend definition of $S(F,G)$ to $S(F,K)$

F is a graph on $\{1, 2, \ldots, k\}$

$$S(F, K) = \int_{[0,1]^k} \prod_{i,j \in e(F)} K(x_i, x_j) \prod_{i=1}^{k} dx_i$$

$\boxed{\text{Same for } t(F, K)}$

- $S(F, K) = \mathbb{E}\left[\prod_{(i,j) \in e(F)} K(U_i, U_j)\right]$

- normalized NUMBER of embeddings of F into a weighted graph (K) with uncountably many vertices. $[0,1]$..

- Uncountable — embeddin / Homomorphism.

[Lovasz - Szegedy 2006]

Theorem:- let $(G_n)_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{F}, d_{sub})$. Then either $(G_n)_{n\geq 1}$ is eventually constant or there is a graphon K such that

$$s(F, G_n) \longrightarrow s(F, K) \quad \forall F \in \mathcal{F}.$$

Remarks:- [LS 06] - t not s

K - not unique. - upto measure preserving transformations.

$K_1 \sim K_2$ if there is a measuring transformation $\sigma : [0,1) \to [0,1]$?

$$K_1(x,y) = K_2(\sigma(x), \sigma(y))$$

$$K_1 \sim K_2 \iff d_{sub}(K_1, K_2) = 0$$

$\mathcal{K}$ = equivalence class of graphs

$(\mathcal{F} \cup \mathcal{K}, d_{sub})$ - Complete metric space & Compact.

- Understand other equivalent metric
  - cut metric e.g.

Postpone discussion on this

For any graph $G \equiv K_G$ - graphon.

Divide $[0,1]$ $\quad I_k = \left[\frac{k}{n}, \frac{k+1}{n}\right)$ $\quad 0 < k < n-2$

$$I_{n-1} = \left[\frac{n-1}{n}, 1\right]$$

Set

$$K_G = \begin{cases} 1 & \text{on } I_m \times I_l \text{ if } (m,l) \in e(G) \\ 0 & \text{otherwise} \end{cases}$$

Ex:- F - fixed graph on k vertices.

$$t(F,G) = S(F, K_G)$$

A graphon $K : [0,1]^2 \to [0,1]$ is said to be finite type if there is a partition of $[0,1]$ into $A_1, A_2, \ldots A_n$ such that K is constant on $A_i \times A_j$.

G - graph Then K_G is of finite type.