Sparse Random Graphs : Assignment 2

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Submit solutions via Moodle by 11th October 10:00 PM.

1. Let G be a Cayley graph with a finite symmetric generating set S such that $o \notin S$ where o is the identity element. Let $B_n = \{v : d(v,o) \leq n\}$. Suppose that $|B_n \setminus B_{n-1}|/|B_n| \to 0$ as $n \to \infty$. Show that $B_n^{(G)}(o) \stackrel{LW}{\to} (G,o)$.

2. Let $G_n = G(n,p_n)$ with $p_n = \lambda/n$. Show that for all $k \geq 1$ and $n \geq \lambda k$, we have that
$$\mathbb{P}(|C(v)| > k) \leq e^{-c_\lambda k},$$
with $c_\lambda > 0$ if $\lambda < 1$.

3. Let $G_n = G(n,p_n)$ with $p_n = 1/n$. Assume that there exists $c < \infty$ such that $\mathbb{E}[|C_n(1)|] \leq cn^{1/3}$ for all n large enough. Show that for all n large enough and $a > 0$,
$$\mathbb{P}(|C_{max}| \geq an^{2/3}) \leq ca^{-2},$$
where $|C_{max}|$ is the size of the largest component in G_n.

4. Let G_n be the tree of depth k in which every vertex except the $3 \times 2^{k-1}$ leaves have degree 3 and where $n = 3(2^k - 1)$. What is the local weak limit of G_n ?

5. Let G_n be a sequence of random graphs on $[n]$ such that $G_n \stackrel{LW-d}{\to} (G,o)$. Let $Z_{\geq k}$ be the number of vertices i such that $|C_n(i)| \geq k$ where $C_n(i)$ is the component of i in G_n. Show that for all $k \geq 1$,
$$n^{-1}\mathbb{E}[Z_{\geq k}] \to \mathbb{P}(|C_G(o)| \geq k),$$
where $C_G(o)$ is the component of the root in (G,o).

6. Let G_n be a sequence of random graphs on $[n]$ such that $G_n \stackrel{LW-d}{\to} (G,o)$. Let $|C_{max}|$ be the size of the largest component in G_n. Assume that $\mathbb{P}(|C_G(o)| = \infty) = 0$, where $C_G(o)$ is the component of the root in (G,o). Show that
$$n^{-1}\mathbb{E}[|C_{max}|] \to 0.$$

ADDITIONAL PROBLEMS (not to be submitted)

1. Let $G_n = G(n, p_n)$ with $p_n = \lambda/n$. Let $B_k = \{i \in [n] : d_{G_n}(i, 1) \leq k\}$, $k \geq 1$. Show that for all $k \geq 1$,

$$\mathbb{E}[|B_k \setminus B_{k-1}|] \leq \lambda^k.$$

Thus conclude that $\mathbb{P}(rad_{G_n}(1) \geq k) \leq e^{-c_\lambda k}$ for $\lambda < 1$ where $rad_G(1) = \max_{v \in C_G(1)} d_G(v, 1)$.

2. Construct the simplest (in your opinion) possible example where the local weak limit of a sequence of deterministic graphs is random.