Recall:- $K : [0,1]^2 \to [0,1]$ Symmetric and measurable

$$G(n,K) := \begin{cases} V = \{1, \dots, n\} & E \quad i \sim j \text{ with probability } K(U_i, U_j) \\ \text{labels} \quad \underset{U_1}{\uparrow} \quad \cdots \quad \underset{U_n}{\uparrow} \sim \text{iid } U[0,1] & \end{cases}$$

Theorem:- Let $\{U_i\}_{i\geq 1}$ be a sequence of i.i.d. Uniform $[0,1]$ random variables. let $K : [0,1]^2 \to [0,1)$ be a graphon, which is continuous a.e on $[0,1] \times [0,1]$.

Then $d_{sub}(G(n,K), K) \xrightarrow[\text{as } n\to\infty]{} 0$ with Probability 1.

Proof:-

- E.T.S. F - finite graph on m-vertices

$$S(F, G(n,K)) \xrightarrow[\text{as } n\to\infty]{} S(F,K) \quad \text{w.p. } 1$$

$$\Rightarrow \quad d_{sub}(G(n,K), K) \to 0 \text{ as } n\to\infty \text{ w.p. } 1.$$

- Condition $(U_1, \dots U_n) = (x_1, \dots x_n) :\equiv x$

$$S(F, G(n,K))\Big|_{(U_1 \dots U_n) = x} := S(F, G(n,x,K))$$

$$\mathbb{P}\left(\left| S(F, G(n,x,K)) - \mathbb{E}[S(F, G(n,x,K))] \right| > \varepsilon \right) \leq 2e^{-a_n}$$

[mc Diarmid's Concentration] $a_n = O(n^2)$.

Borel cantelli $\Longrightarrow$ $S(F, G(n,x,K)) \longrightarrow \mathbb{E}[S(F, G(n,x,K))]$

as $n \to \infty$ with probability one.

Rest of Proof of Theorem :-

$$S(F, G(n,k)) \equiv S(F, G(n, (u_1..u_n), k)$$

shown :-

$$|S(F, G(n, (u_1..u_n), k) - E[S(F, G(n, (u_1..u_n), k)]| \to 0$$

with probability one as $n \to \infty$

Set up Notation :-

$$x = (x_1, .., x_n) \in [0,1]^n \quad ; \quad z_i \in [0,1] \quad 1 \le i \le m$$

F - with m vertices.

$$\mu_F(x) = \frac{1}{(n)_m} \sum_{(i_1,..,i_m)} \underbrace{T_F(x_{i_1}, .., x_{i_m})}$$

where

$(n)_k = n(n-1)..(n-k+1)$ and $\sum_{(i_1..i_m)}$ Sum over range all vectors $(i_1,..,i_m)$ with mutually different coordinates.

$$T_F(z_1, .., z_m) = \prod_{\underbrace{(i,j) \in e(F)}} k(z_i, z_j)$$

claim :-

$$E[S(F, G(n, x, k))] = \mu_F(x). \quad -(\text{Exercise})$$

To finish proof of Theorem :-

$$E[S(F, G(n, u_1..u_n, k)] = \mu_F(u_1 .. u_n) \longrightarrow S(F, k)$$ as $n \to \infty$ wp 1

observations :- $s(F,k) = \int_{[0,1]^k} \prod_{i=1}^{k} dx_i \prod_{(i,j)\in e(F)} k(x_i, x_j)$

$= E\left[\prod_{(i,j)\in e(F)} k(V_i, V_j)\right]$

$V_i \sim U(0,1)$ & independent

To show :

$$\mu_F(U_1, \ldots, U_n) \longrightarrow E\left[\prod_{(i,j)\in e(F)} k(V_i, V_j)\right]$$

as $n \to \infty$ with probability 1

i.e. lemma :-

$$\underbrace{\frac{1}{(n)_m} \sum_{(i_1, \ldots i_m)} T_F(U_{i_1}, \ldots, U_{i_m})}_{\text{Generalised U-statistic}} \xrightarrow[\text{as } n\to\infty]{} E\left[\prod_{(i,j)\in e(F)} k(V_i, V_j)\right]$$

with probability 1

Law of large numbers

where $T_F(z_1 \ldots z_m) = \prod_{(i,j)\in e(F)} k(z_i, z_j)$

Proof :- Define $C_m \equiv \left\{ h : [0,1]^m \to [0,1] \;\middle|\; \begin{array}{l} h \text{ is continuous} \\ \text{a.c.} \end{array} \right\}$

$\forall\, h \in C_m$ let : $Y_n(x) = \dfrac{1}{n^m} \sum_{1 \le i_1 \ldots i_m \le n} h(x_{i_1}, \ldots x_{i_m})$

$x = (x_1 \ldots x_n)$

Step 1 :- $\mathcal{P}_m = \{ h \in C_m \mid \exists\, h_1, \dots, h_m \in C_1 \text{ s.t. } h(\cdot) = \prod_{i=1}^{m} h_i(\cdot) \}$

$h \in \mathcal{P}_m$, $\quad \gamma_h(U_1, \dots, U_n) = \frac{1}{n^m} \sum_{1 \le i_1, \dots i_m \le n} h(U_{i_1}, \dots U_{i_m})$

$$= \frac{1}{n^m} \sum_{1 \le i_1, \dots i_m \le n} \prod_{j=1}^{m} h_j(U_{i_j})$$

$$= \prod_{j=1}^{m} \left[\frac{1}{n} \sum_{l=1}^{n} h_j(U_l) \right]$$

apply SLLN for each j & $U_1, \dots U_n$ are iid $U[0,1]$

$\Rightarrow$

$$\gamma_h(U_1, \dots, U_n) \xrightarrow[\text{as } n \to \infty]{} \mathbb{E}\, h(V_1, \dots V_m)$$

$1 \le i \le m$
V_i are $U(0,1)$
independent

Step 2 :- Approximation : let $\varepsilon > 0$ be given

$\forall\, h \in C_m$: $\exists\, s_1, s_2 \in \text{Span}(\mathcal{P}_m)$ such that

ⓐ $|h - s_1| \le s_2$ a.e (Ex.)

ⓑ $\int s_2(x) \prod_{i=1}^{m} dx_i \le \varepsilon$

— continue — next class —

object & interest $\longrightarrow$ $V_h(U_1, \dots, U_n)$

(i). $V_{S_i}(U_1 \dots U_n) \xrightarrow[\text{step 1}]{} E[S_i(V_1 \dots V_m)]$ as $n \to \infty$ w.p. 1

$S_i \in \operatorname{Span}(B_m)$ $\quad i = 1, 2$

(ii). step2 $\quad E(S_2(V_1 \dots V_m)] \leq \varepsilon$

(i) and (ii) we get: $\exists\, N \geq 1,\ \forall\, n \geq N$

$$|V_{S_1}(U_1 \dots U_n) - E[S_1(V_1 \dots V_m)]| < \varepsilon$$

$$|V_{S_2}(U_1 \dots U_n)| \leq \varepsilon$$

$$|V_h(U_1, \dots U_n) - E\, h(V_1, \dots, V_m)|$$

start from here next class

$$\leq \left\{ \begin{array}{l} |V_h(U_1 \dots U_n) - V_{S_1}(U_1 \dots U_n)| \\ + |V_{S_1}(U_1 \dots U_n) - E S_1(V_1 \dots V_m)| \\ + |E(S_1(V_1 \dots V_m)) - E\, h(V_1 \dots V_m)| \end{array} \right.$$

$$\leq 4\varepsilon$$