6/10/20 : L6 – LWC FOR RANDOM GRAPHS

THM: Let G_n be as above. $G_n \xrightarrow{LW} (G,o)$ iff $\forall\, H_* \in \mathcal{G}_*$ we have that

$$\frac{1}{n}\sum_{u\in V_n} \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_*\right] \to \mathbb{P}\left(B_r^{G}(o) \cong H_*\right)$$

Proof: $\Rightarrow$ as $\mathbb{1}\left[B_r^{(G)}(o) \cong H_*\right]$ is cts. (see A1, Q1)

$\Leftarrow$ (non-trivial & more useful).

Let $h: \mathcal{G}_* \to \mathbb{R}$ be bdd cts & $\varepsilon > 0$.

By Portmanteau theorem, we can assume that h is uniformly cts as well.

So $\exists\, \delta > 0 \ni d((G'',o''),(G',o')) < \delta$

$\Rightarrow |h(G'',o'') - h(G',o')| < \varepsilon/4$

Choose $r = r(\delta) \ni \forall\, (G',o')$

$$d((G',o'), B_r(o')) \le \frac{1}{r+1} < \delta.$$

By Δ ineq,

$$\left|\mathbb{E}_n[h(G_n,o_n)] - \mathbb{E}[h(G,o)]\right|$$
$$\le \left|\mathbb{E}_n[h(B_r^{(G_n)}(o_n))] - \mathbb{E}[h(B_r^{(G)}(o))]\right| + \varepsilon/2.$$

Q: $\exists$ only finitely many graphs of depth r then easy to approximate $h(B_r^{(G_n)}(o_n)) \approx h(B_r^{(G)}(o))$ by our assumption.

One obstruction to have finitely many graphs of depth r is that vertices can have arbitrarily large degrees. But this prob. can be made small.

Define $E_{r,k}(G,o) = \{\exists\, v \in B_r(o) \ni d_v(G) > k\}$

$E_{r,k}(G,o) \downarrow \{\exists\, v \in B_r(o) \ni d_v(G) = \infty\}$ as $k \uparrow \infty$.

$\Rightarrow \forall\, \eta > 0, \exists\, k \ni \mathbb{P}(E_{r,k}(G,o)) \leq \eta \quad (\mathbb{P}(\cdot) = 0)$

Consider $E_{r,k}(G,o)^c$ — $\exists$ fin. many rooted graphs $H_1, \ldots, H_m \ni E_{r,k}(G,o)^c = \bigcup_{i=1}^{m} \{B_r(o) \cong H_i\}$

|||ly $E_{r,k}(G_n,o_n)^c = \bigcup_{i=1}^{m} \{B_r(o_n) \cong H_i\}$

By our assumption, we've that

$$\mathbb{P}_n(B_r(o_n) \cong H_i) \longrightarrow \mathbb{P}(B_r(o) \cong H_i) \quad 1 \leq i \leq m$$

$$\Rightarrow \mathbb{P}_n(E_{r,k}(G_n,o_n)^c) \longrightarrow \mathbb{P}(E_{r,k}(G,o)^c) \quad (*)$$

$\Rightarrow \exists\, n_0 \ni \forall\, n \geq n_0$

$$\mathbb{P}(E_{r,k}(G_n,o_n)) \leq 2\eta$$

$$\Rightarrow \left|\mathbb{E}_n[h(B_r(o_n))] - \mathbb{E}[h(B_r(o))]\right|$$

$$\leq \left|\mathbb{E}_n[h(B_r(o_n))\mathbb{1}_{E_{r,k}(G_n,o_n)^c}] - \mathbb{E}[h(B_r(o))\mathbb{1}_{E_{r,k}(G,o)}]\right| + 2\eta\|h\|_\infty, \quad \forall\, n \geq n_0.$$

Now since $E_{r,k}(G_n, o_n)^c$ is determined by finitely many graphs, from our assumption & (*) we obtain that $\mathbb{E}[h(B_r(o_n))\mathbb{1}_{E_{r,k}(G_n,o_n)^c}]$

$$\longrightarrow \mathbb{E}[h(B_r(o))\mathbb{1}_{E_{r,k}(G,o)^c}].$$

$\Rightarrow \exists\ n_1 \geq n_0 \ni \forall\ n \geq n_1$

$$\left|\mathbb{E}[h(B_r(o_n))\mathbb{1}_{E_{r,k}(G_n,o_n)^c}] - \mathbb{E}[h(B_r(o))\mathbb{1}_{E_{r,k}(G,o)^c}]\right| \leq \varepsilon/4$$

Choosing η small & plugging in the above bounds, the proof is complete. ∎

What about random graphs G_n?

$\mathbb{E}_n[h(G_n,o_n)] = \frac{1}{n}\sum_{i\in[n]} h(G_n, i)$ — random variable.

So how do we interpret convergence?

<u>Defn</u>: $G_n = ([n], E(G_n))$ be a seq. of RGs (possibly disconn.)

(1) $G_n \xrightarrow{LW-d} (G,o)$ if $\mathbb{E}[h(G_n,o_n)] \to \mathbb{E}_{\mu}[h(G,o)]$

$\forall$ bdd cts $h: \mathcal{G}_* \to \mathbb{R}$.

$$\mathbb{E}[h(G_n, o_n)] := \mathbb{E}\left[\mathbb{E}_n[h(G_n, o_n)]\right]$$

Exp. over G_n & o_n.

$$= \frac{1}{n}\mathbb{E}\left[\sum_{i \in [n]} h(G_n, i)\right]$$

(2) $G_n \xrightarrow{LW\text{-}p} (G, o)$ if $\mathbb{E}_n[h(G_n, o_n)] \xrightarrow{p} \mathbb{E}_{\mu}[h(G, o)]$

$\forall$ bdd cts h.

Rem: (1) One can define a.s. convergence as well but not so much of interest to us now. In general, not common.

(2) In LW-p, we have assumed that the limit (G, o) has prob. distribn μ on $\mathcal{G}_*$.

But we can also assume μ is a random prob. measure on $\mathcal{G}_*$ i.e., μ can vary with realizations of G_n.

(Ex (A) : Construct examples).

(3) Our focus will purely be with deterministic limits.

THM: Let $(G_n)_{n\geq 1}$ be a sequence of graphs (random or deterministic)

(a) $G_n \xrightarrow{LW-d} (G,o)$ iff $\forall H_* \in \mathcal{G}_*$,

$$\mathbb{E}\left[p^{(n)}(H_*)\right] = \frac{1}{n}\sum_{u\in[n]} \mathbb{P}\left(B_r^{(G_n)}(u) \cong H_*\right)$$

$$\longrightarrow \mathbb{P}\left(B_r^{(G,o)} \cong H_*\right) \quad (1)$$

$$\left[p^{(n)}(H_*) = \frac{1}{n}\sum_{u\in[n]} \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_*\right]\right]$$

(b) $G_n \xrightarrow{LW-P} (G,o)$ iff $\forall H_* \in \mathcal{G}_*$

$$p^{(n)}(H_*) \xrightarrow{P} \mathbb{P}\left(B_r^{(G,o)} \cong H_*\right) \quad (2)$$

Rmks: (1) If G_n's are deterministic, then $p^{(n)}(H_*) = \mathbb{E}\left[p^{(n)}(H_*)\right]$

$$\Rightarrow G_n \xrightarrow{LW-d} (G,o) \iff G_n \xrightarrow{LW-P} (G,o).$$

(2) Criteria holds for a.s. convergence also.

Proof: Follows from deterministic criteria Thm. (Ex - complete the details). ∎

THM: Let $\mathcal{T}_* \subseteq \mathcal{G}_*$ be a subset of rooted graphs. Let $\mathcal{T}_*(r)$ be rooted graphs in $\mathcal{T}_*$ with depth/height at most r.

Assume (G,o) random graph $\ni \mathbb{P}((G,o) \in \mathcal{T}_*) = 1$.

Let $(G_n)_{n \geq 1}$ be a sequence of graphs (random/det.)

Then $G_n \xrightarrow{LW-d} (G,o)$ if (C1) holds $\forall H_* \in \mathcal{T}_*(r)$ & $r \geq 1$.

Also $G_n \xrightarrow{LWP} (G,o)$ if (C2) holds $\forall r \geq 1$ & $\forall H_* \in \mathcal{T}_*(r)$.

Proof: T.S.T. $\forall H_* \notin \mathcal{T}_*(r)$, $r \geq 1$

$$\mathbb{P}(B_r^{(G_n)}(o_n) \simeq H_*) \longrightarrow 0 = \mathbb{P}(B_r^{(G)}(o) \simeq H_*)$$

$\mathcal{G}_*(r)$ is countable.

$\mathcal{J}_*(r)$ is countable, $\forall \varepsilon > 0 \ \exists m$

& $\mathcal{J}_*(r,m) \subseteq \mathcal{J}_*(r) \ni |\mathcal{J}_*(r,m)| \le m$

$$\ni \ \mathbb{P}\left(B_r^{(G)}(o) \in \mathcal{J}_*(r,m)\right) \ge 1 - \varepsilon.$$

$$\overline{\lim_{n\to\infty}} \mathbb{P}\left(B_r^{(G_n)}(o_n) \notin \mathcal{J}_*(r)\right)$$

$$= 1 - \underline{\lim_{n\to\infty}} \mathbb{P}\left(B_r^{(G_n)}(o_n) \in \mathcal{J}_*(r)\right)$$

$$\le 1 - \underline{\lim_{n\to\infty}} \mathbb{P}\left(B_r^{(G_n)}(o_n) \in \mathcal{J}_*(r,m)\right)$$

$$= 1 - \mathbb{P}\left(B_r^{(G)}(o) \in \mathcal{J}_*(r,m)\right)$$

(since $\mathcal{J}_*(r,m)$ has only finitely many graphs)
& $\mathbb{P}(B_r(o_n) \cong H_*) \to \mathbb{P}(B_r(o) \cong H_*)$
$\forall H_* \in \mathcal{J}_*(r)$.

$$\le \varepsilon$$

$$\Rightarrow \overline{\lim_{n\to\infty}} \mathbb{P}\left(B_r^{(G_n)}(o_n) \notin \mathcal{J}_*(r)\right) = 0 \qquad \blacksquare$$

Ex: Extend the proof for LW-p.

THM: Let $G_n = G(n,p)$, $p = \frac{\lambda}{n}$, $\lambda < \infty$ be the ER random graph. Let $\mathcal{T}_\lambda$ be BGW(Poisson(λ)) offspring distribution. Then

$$G_n \xrightarrow{LW-p} \mathcal{T}_\lambda .$$

Proof: $\mathbb{E}[p^{(n)}(H_*)] = \frac{1}{n} \sum_{u \in [n]} \mathbb{P}(B_r^{(G_n)}(u) \cong H_*)$

($B_r^{(G_n)}(u)$ are identically distributed in $\mathcal{G}_*$)

$$= \mathbb{P}(B_r^{(G_n)}(1) \cong H_*)$$

(shown in prev. classes)

$$\to \mathbb{P}(B_r^{(\mathcal{T}_\lambda)}(\phi) \cong H_*) \text{ if}$$

H_* is a tree.

Choosing $\mathcal{T}_*$ as rooted trees, we can apply the previous theorem to conclude

that $\quad G_n \xrightarrow{\text{Lw-d}} \mathcal{T}_\lambda$.

Let $T \in \mathcal{T}_*(r)$, $r \geq 1$.

$$N_n(T) = n\, p^{(n)}(T) = \sum_{i \in [n]} \mathbb{1}\left[B_r^{(G_n)}(i) \simeq T\right]$$

$$\frac{\mathbb{E}[N_n(T)]}{n} \longrightarrow \mathbb{P}\left(B_r^{(\mathcal{T}_\lambda)}(\phi) \simeq T\right) \quad (*)$$

we want to show

$$\frac{N_n(T)}{n} \xrightarrow{\mathbb{P}} \quad \text{— (1)}$$

STPT

$$\frac{\mathrm{Var}\left(N_n(T)\right)}{\mathbb{E}[N_n(T)]^2} \longrightarrow 0 . \quad \text{— (2)}$$

$$\mathbb{E}[N_n(T)^2] = n\,\mathbb{P}(B_r^{(G_n)}(1) \cong T) + n(n-1)\,\mathbb{P}(B_r^{(G_n)}(1) \cong T,\ B_r^{(G_n)}(2) \cong T)$$

[by expanding $N_n(T)^2$ & using that $(B_r^{(G_n)}(1), B_r^{(G_n)}(2)) \stackrel{d}{=} (B_r^{(G_n)}(i), B_r^{(G_n)}(j))$, $i \neq j$.

We'll show

$$\mathbb{P}(B_r^{(G_n)}(1) \cong T,\ B_r^{(G_n)}(2) \cong T) \to \mathbb{P}(B_r^{(\mathcal{T}_\lambda)}(\phi) \cong T)^2 \quad - ③$$

From ③

$$\frac{\mathrm{Var}(N_n(T))}{n^2} = \frac{\mathbb{P}(B_r^{(G_n)}(1) \cong T)}{n} + \mathbb{P}(B_r^{(G_n)}(1) \cong T,\ B_r^{(G_n)}(2) \cong T) - \mathbb{P}(\downarrow)\,\mathbb{P}(\downarrow).$$

$$\to 0 \Rightarrow ②$$

③ for $r=1$ uses in $\boxed{A1}$.

$$\mathbb{P}(B_r^n(1) \cong T,\ B_r^n(2) \cong T) \qquad T1$$

$$= \mathbb{P}(B_r^n(1) \cong T,\ B_r^n(2) \cong T,\ d_{G_n}(1,2) > 2r)$$

$$+ \mathbb{P}(\qquad\qquad ,\ d_{G_n}(1,2) \le 2r) \qquad T2$$

$$\mathbb{P}(\cdots \quad ,\ d_{G_n}(1,2) \le 2r)$$

$$\le \mathbb{P}(B_r^{(n)}(1) \cong T,\ d_{G_n}(1,2) \le 2r)$$

$$= \mathbb{E}\left[\mathbb{1}[B_r^n(1) \cong T]\ \mathbb{1}[2 \in B_{2r}^n(1)]\right]$$

$$= \frac{1}{n-1} \mathbb{E}\left[\mathbb{1}_{[B_r^n(1) \cong T]} \sum_{j=2}^{n} \mathbb{1}[j \in B_{2r}^n(1)]\right]$$

(since $\mathbb{1}\{j \in B_{2r}^n(1)\}$ is id. distributed)

$$\le \frac{1}{n} \mathbb{E}\left[|B_{2r}^{(n)}(1)|\right]$$

(From A2) $\mathbb{E}\left[\left|B_k^{(n)}(1) \setminus B_{k-1}^{(n)}(1)\right|\right] \le \lambda^k$.

$$\frac{1}{n}\mathbb{E}\left[\left|B_{2r}^{(n)}(1)\right|\right] \le \frac{1}{n}\sum_{k=0}^{2r} \lambda^k \longrightarrow 0.$$

$\Rightarrow$ T2 $\longrightarrow 0$. 4

T1 $= \mathbb{P}\left(B_r^n(1) \cong T, d_{G_n}(1,2) > 2r\right)$

$\mathbb{P}\left(B_r^n(2) \cong T \mid B_r^n(1) \cong T, d_{G_n}(1,2) > 2r\right)$

By 4 & $*$

$$\mathbb{P}\left(B_r^n(1) \cong T, d_{G_n}(1,2) > 2r\right) \longrightarrow \mathbb{P}\left(B_r^{(T_\lambda)}(\phi) \cong T\right)$$

Let $t = |V(T)|$.

$$B_r^{(n)}(2) \mid \left\{B_r^{(n)}(1) \cong T, d_{G_n}(1,2) > 2r\right\} \overset{d}{=} B_r^{G'_{n-t}}(2) \quad (5)$$

where G'_{n-t} is an

ER random graph on $n-t$ vertices with $p = \frac{\lambda}{n}$.

$$\Rightarrow \mathbb{P}\left(B_r^{(G_n)}(2) \cong T \mid \cdots\cdots\right)$$

$$= \mathbb{P}\left(B_r^{(n-t)}(2) \cong T\right) \to \mathbb{P}\left(B_r^{(T_\lambda)}(\phi) \cong T\right)$$

Shows that (T1) $\longrightarrow \mathbb{P}\left(B_r^{(T_\lambda)}(\phi) \cong T\right)^2$

& we have (3)

Explanation for (5):

$$\mathbb{P}\left(B_r^{(G_n)}(2) \cong T \mid B_r^{(G_n)}(1) \cong T,\ d_{G_n}(1,2) > 2r\right)$$

$$= \sum_{\substack{1 \in F \subseteq [n] \\ |F| = k}} \mathbb{P}\left(B_r^{(G_n)}(2) \cong T \mid B_r^{(G_n)}(1) \cong T,\ V_n = F,\ d_{G_n}(1,2) > 2r\right) \times \mathbb{P}\left(V_n = F \mid B_r^{(G_n)}(1) \cong T,\ d_{G_n}(1,2) > 2r\right)$$

$$[V_n = V(B_r^{(G_n)}(1))] \quad - (6)$$

$G_n^F = G_n \setminus F$ — ER graph on $V_n \setminus F$.

$$= \sum_{\substack{1 \in F \subseteq [n] \\ |F| = k}} \mathbb{P}\left(B_r^{(G_n^F)}(2) \cong T \mid \cdots\cdots\right) \mathbb{P}(\cdot \mid \cdots)$$

$$= \sum_{\substack{1 \in F \subseteq [n] \\ |F| = k}} \mathbb{P}(B_r^{(G_n^F)}(2) \cong T)\, \mathbb{P}(\cdots \mid \cdots)$$

$$= \sum_{\cdots} \mathbb{P}(B_r^{(G_{n-t})}(1) \cong T)\, \mathbb{P}(\cdots \mid \cdots)$$

$$(G_n^F \overset{d}{=} G_{n-t})$$

$$= \mathbb{P}(B_r^{(G_{n-t})}(1) \cong T)$$

$$\longrightarrow \mathbb{P}(B_r^{(\mathcal{L}_x)}(\phi) \cong T) .$$

$\blacksquare$