13/10/20 L7 : Summary & the Giant

Summary —

- ER random graphs - sparse / dense regimes
- Thresholds for appearance of subgraphs (A 1.3)
- Our focus on sparse regime $p = \frac{\lambda}{n}$.

  # of trees $= \Theta(n)$.

  # of tree components $= \Theta(n)$ (A 1.4)

  # of k-cycles $\overset{d}{=}$ Poisson $\left(\frac{\lambda^k}{2k}\right)$ (A 1.5)

  # of subgraphs with more than 1 cycle $= O(n^{-2})$.

  $$(\deg(1), \dots, \deg(k)) \xrightarrow{d} (Z_1, \dots, Z_k)$$

  Z_i i.i.d. Poi(λ).

- "Big insight"

  Local structure of ER random graph $\longrightarrow$ Local structure of BGW(Poi(λ)).

  $\Rightarrow$ Gives some understanding of large/giant components

Giant component – Component of size $\Omega(n)$ i.e., cn

– Formalize the above convergence via LWC or B-S convergence on space of rooted graphs.

– 2 notions of convergence of RG

$\rightarrow$ $ER(n, \frac{\lambda}{n}) \xrightarrow{LWC} BGW(Poi(\lambda)) = T_\lambda$

Local functionals of $G(n, \frac{\lambda}{n})$

$\longrightarrow$ Local fnls of T_λ

(Eg. See A1.1)

– What about non-local?

Something can be said

(A2.5 – If no giant in limit, no giant in G_n.
A2.6)

Good estimates for $G(n,p)$ for $\lambda < 1$.

A2.2 – Some estimates for $G(n, \frac{1}{n})$.

A2.3

Can we understand giant better when $\exists$ an ∞ component in the limit?

What other non-local fnls can we study?

GIANT COMPONENT UNDER LWC

(See Section 2.5 of vdH-2 for details & all results)

We'll see some highlights.

THM: (Upper bound on the giant)

Let G_n denote a finite random graph (may be disconn.) $G_n \xrightarrow{LW-P} (G,o)$. $\zeta = \mathbb{P}(|\mathcal{C}(o)|=\infty)$, $\mathcal{C}(o)$ - component of root o. Then $\forall\ \varepsilon>0$

$$P(|\mathcal{C}_{max}| \geq n(\zeta+\varepsilon)) \to 0.$$

Proof: (Sketch - Extension of A2.5, A2.6).

$\mathcal{C}(v)$ - Component of v.

$\mathcal{C}_i$ - i^{th} largest component (ties broken arbitrarily)

$$Z_{\geq k} = \sum_{v\in[n]} \underbrace{\mathbb{1}[|\mathcal{C}(v)|\geq k]}_{h(G_n,v)} \qquad E_n[h(G_n,o_n)] = \frac{Z_{\geq k}}{n}$$

$$G_n \xrightarrow{LW-P} (G,o) \Rightarrow \frac{Z_{\geq k}}{n} \xrightarrow{P} \zeta_{\geq k} := P(|\mathcal{C}(o)|\geq k)$$

$$\{|\mathcal{C}_{max}| \geq k\} = \{Z_{\geq k} \geq k\}$$

If $Z_{\geq k}\geq 1$, $\quad |\mathcal{C}_{max}| \leq Z_{\geq k}$

& $\zeta_{\geq k} \downarrow \zeta = P(|\mathcal{C}(o)|=\infty)$

$$\Rightarrow P(|\mathcal{C}_{max}| \geq n(\zeta+\varepsilon)) \to 0 \qquad \blacksquare$$

When does $n^{-1}|\mathcal{C}_{max}| \xrightarrow{P} \zeta$? [If $\zeta=0$, A2.6]

THM 1. G_n random graph on $[n]$ & $G_n \xrightarrow{LWP} (G, O)$.

Let $\zeta = \mathbb{P}(|\mathcal{C}_G(O)| = \infty) > 0$.

Assume that

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n^2} \mathbb{E}\left[\#\{x, y \in [n] : |\mathcal{C}_G(x)|, |\mathcal{C}_G(y)| \geq k, x \not\leftrightarrow y\}\right] = 0. \quad (*)$$

Then

$$\frac{|\mathcal{C}_{max}|}{n} \xrightarrow{P} \zeta, \quad \frac{|\mathcal{C}_2|}{n} \xrightarrow{P} 0.$$

Necessary & sufficiency of $(*)$ will be in Assignments.

$X_{n,k} = o_{k,P}(1)$ if

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}} \mathbb{P}(|X_{n,k}| > \varepsilon) = 0$$

LEMMA Under assumptions of thm 1,

$$\frac{1}{n^2} \sum_{i \geq 1} |\mathcal{C}_i|^2 \xrightarrow{P} \zeta^2$$

&

$$\frac{1}{n^2} \sum_{i\geq 1} |\mathcal{C}_i|^2 \mathbb{1}[|\mathcal{C}_i| \geq k] = \zeta^2 + o_{k,P}(1)$$

Proof: $\frac{Z_{\geq k}}{n} = \frac{1}{n} \sum_{i \geq 1} |\mathcal{C}_i| \mathbb{1}[|\mathcal{C}_i| \geq k] = \zeta + o_{k,P}(1)$

$$\frac{1}{n^2} \sum_{\substack{i,j \geq 1 \\ i \neq j}} |\mathcal{C}_i| |\mathcal{C}_j| \mathbb{1}[|\mathcal{C}_i| \geq k, |\mathcal{C}_j| \geq k]$$

$$= \frac{1}{n^2} \#\{x, y \in [n] : |\mathcal{C}_G(x)|, |\mathcal{C}_G(y)| \geq k, x \not\leftrightarrow y\}$$

$$\underset{\substack{\text{(Assumption \&} \\ \text{Markov's ineq.)}}}{=} o_{k,P}(1)$$

$$\frac{1}{n^2} \sum_{i \geq 1} |\mathcal{C}_i|^2 \mathbb{1}[|\mathcal{C}_i| \geq k] = \left(\frac{1}{n} \sum_{i \geq 1} |\mathcal{C}_i| \mathbb{1}[|\mathcal{C}_i| \geq k]\right)^2 + o_{k,P}(1)$$

$$= \zeta_\varphi^2 + o_{k,p}(1).$$

$$\frac{1}{n^2}\sum_{i\geq 1}|\zeta_{\varphi i}|^2 = \frac{1}{n^2}\sum_{i\geq 1}|\zeta_{\varphi i}|^2 \mathbb{1}[|\zeta_{\varphi i}|\geq k] + O\left(\frac{k}{n}\right)$$

$$= \zeta_\varphi^2 + o_{k,p}(1).$$

$$\left[\frac{1}{n^2}\sum_{i\geq 1}|\zeta_{\varphi i}|^2 \mathbb{1}[|\zeta_{\varphi i}|<k] \leq \frac{k}{n}\left(\frac{1}{n}\sum_{i\geq 1}|\zeta_{\varphi i}|\right)\right]$$

<u>Proof of THM 1:</u>

Idea of proof: $q_{i,n} := \dfrac{|\zeta_{\varphi i}| \mathbb{1}[|\zeta_{\varphi i}|\geq k]}{\sum_{i\geq 1}|\zeta_{\varphi i}|\mathbb{1}[|\zeta_{\varphi i}|\geq k]}$

$$\sum_{i\geq 1} q_{i,n} = 1 \qquad \sum q_{i,n}^2 = 1 + o_{k,p}(1)$$

(prev. Lemma)

$$\Rightarrow \max_{i\geq 1} q_{i,n} = 1 + o_{k,p}(1)$$

$$\Rightarrow q_{1,n} = 1 + o_{k,p}(1), \quad q_{2,n} = o_{k,p}(1)$$

$$\Rightarrow \frac{|\zeta_{max}|}{n} = \zeta_\varphi + o_{k,p}(1), \quad \frac{|\zeta_2|}{n} = o_{k,p}(1)$$

$$x_i = |\zeta_{\varphi i}|\mathbb{1}[|\zeta_{\varphi i}|\geq k]/n$$

By prev. lemma & LWC-p, w.h.p.

$$\sum_{i\geq 1} x_i \leq \zeta_\varphi + \varepsilon/4 \quad — (1)$$

$$\sum_{i\geq 1} x_i^2 \geq \zeta_\varphi^2 - \varepsilon^2 \quad — (2)$$

[A_n w.h.p, $\mathbb{P}(A_n) \to 1$]

If $x_{1,n} \leq \zeta_1 - \varepsilon$ for some $\varepsilon > 0$.

i.e., $\overline{\lim} \mathbb{P}(x_{1,n} \leq \zeta_1 - \varepsilon) > 0$

$$\wedge|$$

by (1) $\overline{\lim} \mathbb{P}\left(\sum_{i \geq 1} x_{i,n}^2 \leq (\zeta_1 - \varepsilon)^2 + \left(\frac{5\varepsilon}{4}\right)^2\right)$

$$\wedge|$$

$$\overline{\lim} \mathbb{P}\left(\sum_{i \geq 1} x_{i,n}^2 \leq \zeta_1^2 (1-\varepsilon)\right)$$

If ε is small, this contradicts (2)

so $\overline{\lim} \mathbb{P}(x_{1,n} \leq \zeta_1 - \varepsilon) = 0$.

$$\Rightarrow \lim \mathbb{P}(x_{1,n} \geq \zeta_1 - \varepsilon) = 1.$$

We've shown that

$$\lim \mathbb{P}(x_{1,n} \leq \zeta_1 + \varepsilon) = 1.$$

$$\Rightarrow \frac{|C_{max}|}{n} \xrightarrow{P} \zeta_1.$$

$$\sum_{i \geq 1} x_i^2 = \zeta_1 + o_{\mathbb{P}}(1)$$

$$\& \quad x_1 = \zeta_1 + o_{\mathbb{P}}(1)$$

$\Rightarrow x_2 = o_{\mathbb{P}}(1)$. [Fill the gap!]

A sequence $(X_n)_{n \geq 1}$ of r.v. is called uniformly integrable (UI) if

$$\lim_{K \to \infty} \overline{\lim_{n \to \infty}} \mathbb{E}\left[|X_n| \mathbb{1}\{|X_n| > K\}\right] = 0.$$

If $\sup_{n \geq 1} \mathbb{E}[|X_n|^2] < \infty$

$$\mathbb{E}\left[|X_n| \mathbb{1}[|X_n|>K]\right] \overset{CS}{\leq} \sqrt{\mathbb{E}[X_n^2]} \sqrt{\mathbb{P}(|X_n|>K)}$$

$$\leq \frac{1}{K} \mathbb{E}[X_n^2]$$

THM 2: Same assumptions as in THM 1.

$v_\ell(\mathcal{C}_{max})$ = # of vertices of degree ℓ in $\mathcal{C}_{max}$

$$\frac{v_\ell(\mathcal{C}_{max})}{n} \xrightarrow{p} \mathbb{P}(|\mathcal{C}(o)| = \infty, d_o = \ell)$$

d_o = deg of o in (G, o)

If $\{d_{o_n}^{(n)}\}_{n\geq 1}$ is U.I. then

# of edges in $\mathcal{C}_{max}$ — $\frac{E(\mathcal{C}_{max})}{n} \to \frac{1}{2} \mathbb{E}\left[\mathbb{1}[|\mathcal{C}_o(o)| = \infty] \times d_o\right]$

THM 3: Let $G_n = G(n, p)$, $p = \frac{\lambda}{n}$.

Then

$$\frac{|\mathcal{C}_{max}|}{n} \xrightarrow{p} \zeta_\lambda, \quad \frac{|\mathcal{C}_2|}{n} \xrightarrow{p} 0$$

where $\zeta_\lambda = 1 - p_{ext}(\lambda)$ for BGW(Poi(λ))

$$\frac{v_\ell(\mathcal{C}_{max})}{n} \xrightarrow{p} e^{-\lambda} \frac{\lambda^\ell}{\ell!} \left[1 - p_{ext}(\lambda)^\ell\right]$$

$$\frac{E(\mathcal{C}_{max})}{n} \xrightarrow{p} \frac{1}{2} \lambda \left[1 - p_{ext}(\lambda)^2\right]$$

Proof sketch: We'll verify Assumption in THM 1 for Erdos-Renyi RG.
Since $G_n \xrightarrow{d} (\mathcal{T}_\lambda, \phi)$ & so from thm 2, we've to compute

$$\mathbb{P}(|\mathcal{T}_\lambda| = \infty,\ d_\phi = l) = ?$$

$$\mathbb{E}[d_0 \, \mathbb{1}[|\mathcal{T}_\lambda| = \infty]] = ?$$

If $d_0 = l$ & $|\mathcal{T}_\lambda| = \infty$ then one of the 'l' subtrees at child of ϕ is infinite.
Subtree survival is indep of d_0.
Complete the proof..... ☒

$\therefore \Longrightarrow$ For ER graph, $\exists !$ giant component for $\lambda > 1$ & else no giant component.