2017 : L 8 − Structure of Giant

THM 2: Same assumptions as in THM 1.

$v_\ell(\mathcal{C}_{max})$ = # of vertices of degree ℓ in $\mathcal{C}_{max}$

$$\frac{v_\ell(\mathcal{C}_{max})}{n} \xrightarrow{P} \mathbb{P}(|\mathcal{C}_o(o)|=\infty, d_o=\ell)$$

d_o = deg of o in (G,o)

If $\{d_{o_n}^{(n)}\}_{n\geq 1}$ is U.I. then

# of edges in $\mathcal{C}_{max}$ − $$\frac{E(\mathcal{C}_{max})}{n} \longrightarrow \frac{1}{2}\mathbb{E}\left[\mathbb{1}[|\mathcal{C}_o(o)|=\infty] \times d_o\right]$$

Proof: $A \subseteq \mathbb{N}$. d_v − $\deg(v)$ in G_n

$$Z_{A,\geq k} = \sum_{v\in[n]} \mathbb{1}[|\mathcal{C}_o(v)|\geq k, d_v\in A]$$

$$G_n \xrightarrow{LW-P} (G,o) \Rightarrow \frac{Z_{A,\geq k}}{n} \xrightarrow{P} \mathbb{P}(|\mathcal{C}_o(o)|\geq k, d_o\in A)$$ ↳ (1)

Since $\mathbb{P}(|\mathcal{C}_{max}|\geq k) \longrightarrow 1 \quad \forall\ k\geq 1$

('cos $n^{-1}|\mathcal{C}_{max}| \xrightarrow{P} \zeta > 0$)

$$\mathbb{P}\left(\frac{1}{n}\sum_{a\in A} v_a(\mathcal{C}_{max}) \leq \frac{Z_{A,\geq k}}{n}\right) \longrightarrow 1. \quad \text{− (2)}$$

$A = \{\ell\}^c$. $v_a(\mathcal{C}_{max}) = \#\{v\in\mathcal{C}_{max} : d_v = a\}$

$$P\left(\frac{1}{n}[|\mathcal{C}_{max}| - v_\ell(\mathcal{C}_{max})] \leq \mathbb{P}(|\mathcal{C}_o(o)|\geq k, d_o\neq\ell) + \varepsilon/2\right) \longrightarrow 1. \quad \text{− (3)}$$

From (1) & (2)

$$\Rightarrow \mathbb{P}\left(\frac{v_\ell(\mathcal{C}_{max})}{n} \geq \mathbb{P}(|\mathcal{C}_o(0)| = \infty) - \mathbb{P}(|\mathcal{C}_o(0)| \geq k, d_0 \neq \ell) - \varepsilon/2 \right) \to 1.$$

$$\frac{|\mathcal{C}_{max}|}{n} = \frac{1}{n}\left[|\mathcal{C}_{max}| - v_\ell(\mathcal{C}_{max}) \right] + v_\ell(\mathcal{C}_{max})$$

(using ③) $\leq \mathbb{P}(|\mathcal{C}_o(0)| \geq k, d_0 \neq \ell) + v_\ell(\mathcal{C}_{max}) + \varepsilon/2$

The above event happens w.p. $\to 1$.

Also choose k large $\ni$ $\mathbb{P}(|\mathcal{C}_o(0)| \geq k, d_0 \neq \ell) - \mathbb{P}(|\mathcal{C}_o(0)| = \infty, d_0 \neq \ell) \leq \varepsilon$.

Further, Thm 1 $\Rightarrow$ $\mathbb{P}\left(\frac{|\mathcal{C}_{max}|}{n} \geq \zeta_\varphi - \varepsilon \right) \to 1$

where $\zeta_\varphi = \mathbb{P}(|\mathcal{C}_o(0)| = \infty) > 0$ (by assumption)

$$\Rightarrow \mathbb{P}\left(\frac{v_\ell(\mathcal{C}_{max})}{n} \geq \zeta_\varphi - \mathbb{P}(|\mathcal{C}_o(0)| = \infty, d_0 \neq \ell) - \cancel{\varepsilon}/2 \right) \to 1$$

$$\Rightarrow \mathbb{P}\left(\frac{v_\ell(\mathcal{C}_{max})}{n} \geq \mathbb{P}(|\mathcal{C}_o(0)| = \infty, d_0 = \ell) - \varepsilon/\cancel{2} \right) \to 1$$

Also use ② for $A = \{\ell\}$ & noting that it holds $\forall k$,

$$\mathbb{P}\left(\frac{v_\ell(\mathcal{C}_{max})}{n} \leq \mathbb{P}(|\mathcal{C}_o(0)| = \infty, d_0 = \ell) + \varepsilon \right) \to 1$$

$$\Rightarrow \frac{v_\ell(\mathcal{C}_{max})}{n} \xrightarrow{P} \mathbb{P}(|\mathcal{C}_o(0)| = \infty, d_0 = \ell).$$

$$|E(\mathcal{C}_{max})| = \frac{1}{2} \sum_{\ell \geq 1} \ell \, v_\ell(\mathcal{C}_{max}) = \frac{1}{2} \sum_{v \in [n]} d_v$$

$$\frac{|E(\mathcal{C}_{max})|}{n} = \underbrace{\frac{1}{2n} \sum_{\ell \le K} \ell\, v_\ell(\mathcal{C}_{max})}_{T1} + \underbrace{\frac{1}{2n} \sum_{\ell > K} \ell\, v_\ell(\mathcal{C}_{max})}_{T2}$$

$$T1 \xrightarrow{P} \frac{1}{2} \sum_{\ell \le K} \mathbb{P}(|\mathcal{C}(0)| = \infty, d_0 = \ell)\, \ell$$

$$= \frac{1}{2} \mathbb{E}\left[d_0 \, \mathbb{1}[|\mathcal{C}(0)| = \infty, d_0 \le K]\right]$$

$$n_\ell = v_\ell(G_n) = \#\{v \in [n] : d_v = \ell\} \ge v_\ell(\mathcal{C}_{max})$$

$$\Rightarrow T2 \le \frac{1}{2} \sum_{\ell > K} \frac{\ell\, n_\ell}{n} = \mathbb{E}_n\left[\frac{d_{o_n}^{(n)}}{2} \mathbb{1}[d_{o_n}^{(n)} > K]\right]$$

o_n – random root in G_n

By U.I., $\forall \varepsilon > 0$,

$$\lim_{K\to\infty} \overline{\lim_{n\to\infty}} \mathbb{P}\left(\mathbb{E}_n\left[d_{o_n}^{(n)} \mathbb{1}[d_{o_n}^{(n)} > K]\right] > \varepsilon\right)$$

(Markov's ineq.)
$$\le \lim_{K\to\infty} \overline{\lim_{n\to\infty}} \frac{\mathbb{E}\left[\mathbb{E}_n\left[d_{o_n}^{(n)} \mathbb{1}[d_{o_n}^{(n)} > K]\right]\right]}{\varepsilon}$$

$$\le \lim_{K\to\infty} \overline{\lim_{n\to\infty}} \varepsilon^{-1} \mathbb{E}\left[d_{o_n}^{(n)} \mathbb{1}[d_{o_n}^{(n)} > K]\right] = 0 \quad \text{(by U.I.)}$$

Complete proof by letting $K \uparrow \infty$ in T1. ∎

Remains to prove for $G(n,p)$ —

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n^2} \mathbb{E}\left[\#\{x, y \in [n] : |\mathcal{C}(x)|, |\mathcal{C}(y)| \ge k,\ x \not\leftrightarrow y\}\right] = 0. \qquad (*)$$

LEMMA: Assume other conditions than (*) in THM 1.
Then (*) holds if

(i) $$\lim_{r\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n^2} \mathbb{E}\left[\#\{x,y\in[n] : |\partial B_r(x)|, |\partial B_r(y)| \geq r, x \not\leftrightarrow y\}\right] = 0$$

(ii) $\exists$ $r = r_k \to \infty$ $\ni$ (G,o) satisfies the following -

$$\mathbb{P}(|\mathcal{C}_G(o)| \geq k, |\partial B_r(o)| < r_k) \longrightarrow 0$$
$$\mathbb{P}(|\mathcal{C}(o)| < k, |\partial B_r(o)| \geq r_k) \longrightarrow 0.$$

Rmk: If $r_k = k$, $\mathbb{P}(|\mathcal{C}(o)| < k, |\partial B_r(o)| \geq r) = 0$.

For BW tree with mean $\lambda > 1$,

Ex (A). $$\lambda^{-r} \mathbb{E}[|\partial B_r(\phi)| \mid |\mathcal{C}(\phi)| = \infty] \xrightarrow{a.s.} M$$
for $M > 0$.

Conditioned on survival, $|\partial B_r(\phi)|$ grows exp. fast!
$\Longrightarrow$ first part of (ii) holds.

Note $$\lambda^{-r} \mathbb{E}[|\partial B_r(\phi)|] = 1.$$

Proof: $$P_r^{(2)} = \#\{x,y : |\partial B_r(x)|, |\partial B_r(y)| \geq r, x \not\leftrightarrow y\}$$
$$Q_k^{(2)} = \#\{x,y : |\mathcal{C}(x)|, |\mathcal{C}(y)| \geq k, x \not\leftrightarrow y\}.$$

$$|P_r^{(2)} - Q_k^{(2)}| \leq 2n(Z_1 + Z_2)$$

$$Z_1 = \#\{v : |\partial B_r(v)| < r, |\mathcal{C}(v)| \geq k\}$$

$$Z_2 = \#\{v : |\partial B_r(v)| \geq r, |\mathcal{C}(v)| < k\}$$

$$G_n \xrightarrow{LW-p} (G, \rho) \implies$$

$$\frac{Z_1 + Z_2}{n} \xrightarrow{p} \mathbb{P}(|\mathcal{C}(0)| \geq k, |\partial B_r(0)| < r) + \mathbb{P}(|\mathcal{C}(0)| < k, |\partial B_r(0)| \geq r) \quad -①$$

Note that

$$\frac{1}{n^2} |P_r^{(2)} - Q_k^{(2)}| \leq 2 \frac{(Z_1 + Z_2)}{n}$$

$$\frac{Z_1}{n} \leq 1, \quad \frac{Z_2}{n} \leq 1 \quad \& \quad \frac{Z_1 + Z_2}{n} \xrightarrow{p} \cdots$$

By DCT for convergence in prob. / measure, we have that $\overline{\lim_{n \to \infty}} \frac{\mathbb{E}[|P_{r_k}^{(2)} - Q_k^{(2)}|]}{n^2}$

$$\leq 2 [\text{RHS of } ①]$$

Now by Assumption (ii),

$$\lim_{k \to \infty} \overline{\lim_{n \to \infty}} \frac{\mathbb{E}[|P_{r_k}^{(2)} - Q_k^{(2)}|]}{n^2} = 0$$

$$\lim_{k \to \infty} \overline{\lim_{n \to \infty}} \frac{\mathbb{E}[Q_k^{(2)}]}{n^2} \leq \lim_{k \to \infty} \overline{\lim_{n \to \infty}} \frac{\mathbb{E}[P_{r_k}^{(2)}]}{n^2}$$

$$= 0$$

(by Assumption (i))

■

Let O_1, O_2 be 2 vertices chosen uniformly at random & indep. in $[n]$.

$$\frac{1}{n^2}\mathbb{E}\left[\#\{x,y : |\partial B_r(x)|, |\partial B_r(y)| \geq r,\ x \not\leftrightarrow y\}\right]$$

$$= \mathbb{P}\left(|\partial B_r(O_1)|, |\partial B_r(O_2)| \geq r,\ O_1 \not\leftrightarrow O_2\right)$$

$$= \sum_{\substack{b_0^1, b_0^2 \geq r \\ s_0^1, s_0^2}} \mathbb{P}\left(O_1 \not\leftrightarrow O_2,\ |\partial B_r(O_i)| = b_0^i,\ |B_{r-1}(O_i)| = s_0^i,\ i=1,2\right)$$

$\partial B_r(O_i)$ — vertices having an edge to a vertex in $B_{r-1}(O_i)$ but not connected to O_2

$$|\partial B_r(O_1)| \leq \mathrm{Bin}\left(\underbrace{n - s_0^{(1)} - s_0^{(2)}}_{\substack{\text{choice of vertices} \\ \text{available to } B_{r-1}(O_i)}},\ p\right)$$

large boundary of O_1 $\Rightarrow$ more choices for edges

$\Rightarrow$ less likely to miss a vertex in $\mathscr{C}(O_2)$ if $\partial B_r(O_2)$ is large.

See vdH Vol. 2, Sec 2.5. for more details.

$$\mathbb{P}(d_{G_n}(o_1, o_2) \le k) = \mathbb{E}[\mathbb{1}[o_2 \in B_k(o_1)]]$$

o_2 is random vertex

F-T theorem

$$= \mathbb{E}\Big[\frac{1}{n}\sum_{v\in[n]} \mathbb{1}[v \in B_k(o_1)]\Big]$$

$$= \frac{1}{n}\mathbb{E}[|B_k(o_1)|]$$

$$= \frac{1}{n}\mathbb{E}[|B_k(1)|]$$

$$= \frac{1}{n}\sum_{l=0}^{k}\lambda^l = \frac{\lambda^{k+1}-1}{n(\lambda-1)}$$

for $\lambda > 1$.

If $k = \dfrac{(1-\varepsilon)\log n}{\log \lambda}$

then $\mathbb{P}\Big(d_{G_n}(o_1, o_2) \le \dfrac{(1-\varepsilon)\log n}{\log \lambda}\Big) \longrightarrow 0.$

$\Rightarrow$ Two random vertices are at distance $\ge \log_\lambda n$

THM (THM 2.26 of vd H vol. 2)

Conditioned on $o_1 \longleftrightarrow o_2$,

$$\frac{d_{G_n}(o_1, o_2)}{\log n} \xrightarrow{P} \frac{1}{\log \lambda}$$