Graphon: $K : [0,1]^2 \longrightarrow [0,1]$ symmetric and measurable.

F - graph on m vertices

$$S(F, K) = E\Big[\prod_{(i,j)\in e(F)} K(U_i, U_j)\Big] \qquad U_i \stackrel{d}{=} U[0,1] \text{ r.v's}$$

$$= \int \prod_{i=1}^{m} dx_i \prod_{(i,j)\in e(F)} K(x_i, x_j)$$

Finite graphs from K :- $G(n, K)$ $V = \{1, \dots, n\}$

$i \sim j$ w.p $K(U_i, U_j)$ where $U_1, \dots, U_n$ i.i.d $U[0,1]$

$$d_{sob}(G(n,K), K) \longrightarrow 0 \quad a.s \quad \text{as } n \to \infty$$

Key :- $\forall$ F, $|F| = m$

$$S(F, G(n,K)) \longrightarrow S(F, K) \quad \text{as } n \to \infty$$

w.p 1

Question: K_1, K_2 two graphons.

$n \geq 1$ — Construct $G(n, K_1)$, $G(n, K_2)$

Can we tell K_1 and K_2 are different from $G(n, K_1)$ & $G(n, K_2)$?

Lemma: [BCLSV 2008, lemma 4.4] :-

F – graph on n-vertices

$$\mathbb{P}\left(|s(F, G(n,K)) - s(F,K)| > \varepsilon \right) \leq \exp\left(-\frac{\varepsilon^2 n}{4m^2} \right)$$

Classical – revision of Hypothesis testing

Q: A coin when tossed 1000 times produced 550 heads. Is the coin fair ?

A:-

$$X_i = \begin{cases} 1 & \text{if the } i\text{th toss Head} \\ 0 & \text{if the } i\text{th toss tail} \end{cases}$$

$\Rightarrow$ X_i are independent, Bernoulli(p)

Is $p = \frac{1}{2}$?

$$S_n = \sum_{i=1}^{n} X_i \qquad\qquad \underline{\underline{S_{1000} = 550}}$$

$$S_n \sim \text{Binomial}(n, p)$$

<u>Central limit Theorem</u> :

$$\frac{S_n - np}{\sqrt{np(1-p)}} \xrightarrow{d} N(0,1)$$

$$\boxed{\begin{array}{l}\text{To address} \\ p = \frac{1}{2} \text{ or} \\ p \neq 1/2\end{array}}$$

Suppose $p = 1/2$ [Null hypothesis]

$$\frac{S_n - \frac{n}{2}}{\sqrt{n\, 1/4}} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty$$

<u>$n = 1000$</u>

$$\mathbb{P}(S_{1000} \geq 550) = \mathbb{P}\left(\frac{S_{1000} - 500}{\sqrt{250}} \geq \frac{50}{\sqrt{250}} \right)$$

$$\simeq \mathbb{P}\left(Z \geq \frac{50}{\sqrt{250}} \right) \text{ — Compute this number}$$

Suitably one can decide if we accept / reject the Hypothesis that $p = 1/2$

$$\frac{S_n}{n} \xrightarrow{\text{w.p.1}} p \quad \text{as } n \to \infty \qquad \text{[SLLN]}$$

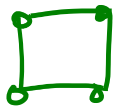
$$\frac{S_n - np}{\sqrt{np(1-p)}} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty \qquad \text{[C.L.T.]}$$

Special case : $k : [0,1]^2 \to [0,1]$, $k(x,y) = p$ | $G(n,k) \equiv G(n,p)$ Erdös - Renyi

Ex:

$m=2$, $F_2 =$ 

$S(F_2, G(n,k)) \longrightarrow p$. as $n \to \infty$

$m=4$, $F_4 =$ 

$S(F_4, G(n,k)) \longrightarrow p^4$ as $n \to \infty$

Theorem [C, G, W -89] G_n be a dense graph sequence (non-constant) such that

$$S(F_2, G_n) \longrightarrow p \;,\quad S(F_4, G_n) \longrightarrow p^4$$

for some $0 < p < 1$ then $\{G_n\}_{n \geq 1}$ converge in $d_{sub}(\cdot\,;\,)$ metric & limiting graphon $k \equiv p$.

Q:- Can one establish a C.L.T. for the two statistics $S(F_2, G_n)$ & $S(F_4, G_n)$ to test for p?

$\tilde{Q}$:- $k \equiv p$ can one show

Ex

$$\frac{S(F_2, G(n,k)) - \overset{*}{=} p}{\sqrt{Var(S(F_2, G(n,k)))}} \xrightarrow{d} N(0,1) \ ?$$

$\mathcal{L}(X) :\equiv$ distribution of the random variable X.

$$d(\mathcal{L}(X), \mathcal{L}(Y)) = \sup_{A \in \mathcal{B}} |P(X \in A) - P(Y \in A)|$$

$$T_1(G_n) = \frac{|\mathrm{inj}(F_2, G_n)|}{2}, \quad T_2(G_n) = \frac{|\mathrm{inj}(F_4, G_n)|}{8}$$

$2 = \# aut(F_2)$ $\quad$ $8 = \# aut(F_4)$

$\Rightarrow T_1 = \#$ of edges in G_n $\qquad E[T_1] = {}^nC_2\, p$

$T_2 = \#$ of 4-cycles in G_n

$$Var(T_1) = {}^nC_2\, p(1-p)$$

$Var(T_2) = \ldots$ Ex $\qquad Cov(T_2, T_1) = \ldots$

Thm [R, X 11] :

① $$\frac{T_1(h_n) - \binom{n}{2}p}{\sqrt{\binom{n}{2}p(1-p)}} = S_n$$

- $S_n \xrightarrow{d} N(0,1)$
- $d(\mathcal{L}(S_n), \mathcal{L}(N(0,1))) \leq \frac{C}{p^3(1-p)^3\sqrt{n}}$

② $$\frac{T_2(h_n) - *_n}{\sqrt{(**)_n}} \equiv \tilde{S}_n$$

$$d(\mathcal{L}(\tilde{S}_n), \mathcal{L}(N(0,1))) \leq \frac{C}{p^3(1-p)^3\sqrt{n}}$$

③ $$d(\mathcal{L}(S_n, \tilde{S}_n), \mathcal{L}(2,2)) \leq \frac{C}{p^3(1-p)^3\sqrt{n}}$$

$$K = \begin{array}{|c|c|}\hline \delta & \beta \\ \hline \alpha & \delta \\ \hline \end{array}$$

γ

$k(x,y) \equiv$ block-model

$G(n,k)$ $\quad u_1 \dots u_n \quad U[0,1)$

$i \sim j$ w.p. $k(u_i, u_j)$

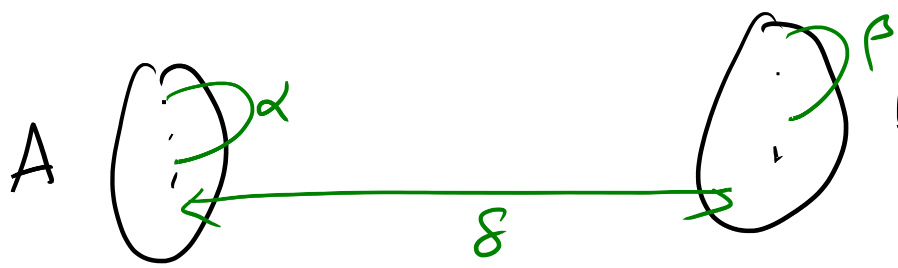
Finitely determinable graphons

$A = \{ k : u_k < \gamma \}$ $\qquad A \cap B = \phi$

$B = \{ l : u_l > \gamma \}$

$\frac{A}{A+B} = \gamma$

$\frac{B}{A+B} = 1-\gamma$



Theorem [Janson 1991] — complete description next class

$\{X_i\}_{i=1}^{n}$ $\quad \{Y_{ij}\}_{1 \le i < j \le n}$ — two independent sequences

$$S_{m,n}(f) = \sum_{i_1 < \dots < i_m \le n} f(X_{i_1}, \dots, X_{i_m}, Y_{i_1 i_2}, \dots, Y_{i_{m-1}, i_m})$$

$$\frac{S_{m,n}(f) - *}{\sqrt{**}} \xrightarrow{d} N(0,1)$$