Sparse Random Graphs : Assignment 3

Yogeshwaran D.

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Submit solutions via Moodle by 15th November 10:00 PM.

1. Let G_n converge in probability in the local weak sense to (G, o). Let $(o_n^{(1)}, o_n^{(2)})$ be two independent uniformly chosen vertices in $[n]$. Show that $((G_n, o_n^{(1)}), (G_n, o_n^{(2)})) \xrightarrow{d} ((G, o), (G', o'))$ where (G', o') is an independent copy of (G, o).

2. Construct an example where G_n converges in probability in the local weak sense to (G, o), while $\frac{|C_{max}|}{n} \xrightarrow{p} \eta < \zeta = \mathbb{P}(|C(o)| = \infty)$.

3. Let G_n be a finite (possibly disconnected) random graph and converge in probability in the local weak sense to (G, o). Let $\zeta = \mathbb{P}(|C(o)| = \infty)$. Assume that

$$\limsup_{k\to\infty} \limsup_{n\to\infty} n^{-2}\mathbb{E}[|\{x, y \in [n] : |C(x)|, |C(y)| \geq k, x \notin C(y)\}|] > 0.$$

   Then prove that for some $\epsilon > 0$,

$$\limsup_{n\to\infty} \mathbb{P}(|C_{max}| \leq n(\zeta - \epsilon)) > 0.$$

4. Under the assumptions of Q.3., show that for some $\epsilon > 0$,

$$\limsup_{n\to\infty} \mathbb{P}(|C_2| \geq \epsilon n) > 0,$$

   where C_2 is the second largest component with ties broken arbitrarily.

5. The *size-biased version* X^* of a non-negative random variable X is defined as

$$\mathbb{P}(X^* \leq x) = \frac{\mathbb{E}[X 1[X \leq x]]}{\mathbb{E}[X]}.$$

   Show that when $(d_n^{o_n})_{n\geq 1}$ forms a uniformly integrable sequence of random variables, there exists a subsequence along which D_n^*, the size-biased version of $D_n = d_n^{o_n}$, converges in distribution. Here G_n is a sequence of

6. Let $G_n = G(n, p)$ for $p = \lambda/n$. Using direct computations (i.e., not using local weak convergence), show that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}(d_{G_n}(o_n^{(1)}, o_n^{(2)}) \leq \frac{\log n}{\log \lambda} - K) = 0,$$

   where $o_n^{(1)}, o_n^{(2)}$ are independent uniformly chosen vertices in $[n]$.