10/11/20: L10 – UNIMODULARITY

$G_n \xrightarrow{LW-} (G, o)$. G_n seq. of finite (random) graphs.

What can we say about set of limit points (G, o)?

What about the root in relation to the graph?

Root has to be "the typical" vertex of the graph.

What is a typical vertex?

Given a finite graph, typical vertex = uniformly chosen?

What abt. infinite graphs?

PROPOSITION: A finite rooted random graph (G, o) has root uniformly distributed iff

$$\mathbb{E}\left[\sum_{v \in V} f(G, o, v)\right] = \mathbb{E}\left[\sum_{v \in V} f(G, v, o)\right] \quad -(1)$$

(doubly rooted.)

$\forall$ non-neg. fns $f(G, u, v)$ of the graph & $u, v \in V$.

Proof: o – Unif rooted.

$$\mathbb{E}\left[\sum_{v \in V} f(G, o, v)\right] = \sum_{v \in V} \mathbb{E}\left[\frac{1}{|V|} \sum_{u \in V} f(G, u, v)\right]$$

(Fubini Tonelli)

$$= \sum_{u \in V} \mathbb{E}\left[\frac{1}{|V|} \sum_{v \in V} f(G, u, v)\right]$$

$$= \sum_{u \in V} \mathbb{E}\left[f(G, u, o)\right]$$

$$= \mathbb{E}\left[\sum_{u \in V} f(G, u, 0)\right]$$

$$\Rightarrow \text{①}.$$

Converse: Define $f(G, x, y) = \dfrac{h(G, x)}{|V|}$

$h : \mathcal{G}^* \longrightarrow \mathbb{R}_+$.

$$\mathbb{E}\left[\sum_{v \in V} f(G, 0, v)\right] = \mathbb{E}\left[h(G, 0)\right]$$

$$\parallel$$

$$\mathbb{E}\left[\sum_{v \in V} f(G, v, 0)\right] = \mathbb{E}\left[\sum_{v \in V} \frac{h(G, v)}{|V|}\right]$$

$$\Rightarrow \mathbb{E}\left[h(G, 0)\right] = \frac{1}{|V|} \sum_{v \in V} \mathbb{E}\left[h(G, v)\right]$$

$$\Rightarrow 0 \stackrel{d}{=} \text{Unif}(V).$$

■

Interpretation of ①

$f(G, u, v)$ — Mass sent from u to v in G.

$\sum_{v \in V} f(G, u, v)$ — Total outgoing mass at u

$\sum_{v \in V} f(G, v, u)$ — " incoming " at u

MTP — Mass transport principle (i.e., ①).

(Ignores graph structure).

$\Rightarrow$ $\mathbb{E}$[outgoing mass at root]

$= \mathbb{E}$[incoming mass at root].

Uniform root in fin. graphs $\equiv$ MTP

∞ graphs ?

$\mathcal{G}_{**}$ – space of doubly rooted loc. finite graphs / $\cong$

$(G, u, v) \cong (G', u', v')$ if

$\exists$ a graph isomorphism $\phi : G \to G'$

$\ni \phi(u) = u', \ \phi(v) = v'$.

So $f : \mathcal{G}_{**} \to \mathbb{R} \Rightarrow f(G, u, v) = f(G', u', v')$

if $(G, u, v) \cong (G', u', v')$.

<u>DEFN</u> $(G, o) \in \mathcal{G}^*$, random graph. We say (G, o) satisfies MTP if

$$\mathbb{E}\left[\sum_{v \in V} f(G, o, v)\right] = \mathbb{E}\left[\sum_{v \in V} f(G, v, o)\right]$$

$\forall f : \mathcal{G}_{**} \to \mathbb{R}_+$.

(G, o) called unimodular if (G, o) satisfies MTP.

$\to$ Restriction to $\mathcal{G}_{**}$ is necessary (Ex.)

$\to$ finite (G, o) is unimodular if o is uniform.

PROPN: $(G,o) \in \mathcal{G}^*$ random graph is unimodular iff it satisfies MTP $\forall f: \mathcal{G}_{**} \to \mathbb{R}_+$ $\ni f(G,x,y) = 0$ if $x \not\sim y$ in G.

[INVOLUTION INVARIANT].

Proof later. i.e., check MTP on fns that are non-trivial only on edges & you get full MTP.

Eg: (1) $(G(n,p), 1)$ is unimodular.

$$\iff E\Big[\sum_{i \neq 1} h(G_n, 1, i)\Big] = E\Big[\sum_{i \neq 1} h(G_n, i, 1)\Big] \quad \text{(check)}$$

$\forall h: \mathcal{G}_{**} \to \mathbb{R}_+$.

Let G be a Cayley graph. Is (G,o) unimodular?

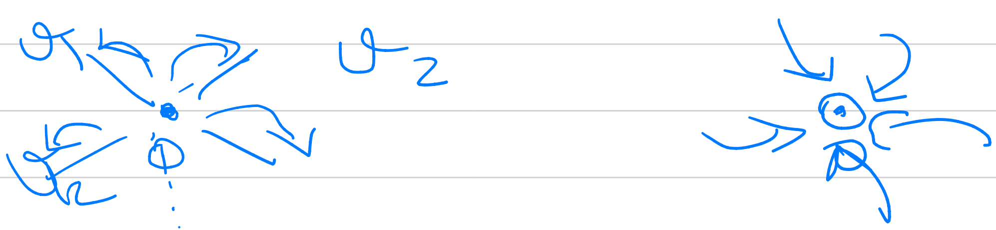
G is <u>transitive</u> if $(G,u) \cong (G,v)$ $\forall u,v \in V$.

G is <u>symmetric</u> if $(G,u,v) \cong (G,u',v')$ $\forall u,v,u',v' \in V \ni u \sim v$ & $u' \sim v'$.

A connected symmetric graph is transitive. (Prove!)

Propn: A conn. symmetric graph G with an arbitrary root is unimodular.

why?



Proof:

For $o \sim v$, $f(G,o,v) = f(G,v,o)$ by symmetry — (2) & $f: \mathcal{G}_{**} \to \mathbb{R}$.

Set $f: \mathcal{G}_{**} \to \mathbb{R} \ni f(G,u,v) = 0$ if $u \not\sim v$.

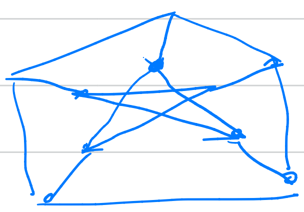
(2) $\Rightarrow$ if $o \sim v$ $\quad f(G,o,v) = f(G,v,o)$.

$$\Rightarrow \sum_{v \in V} f(G,o,v) = \sum_{v \in V} f(G,v,o)$$

By Propn on II, (G,o) is unimodular. ☒

→ Cayley graphs are conn. & transitive. Symmetric? Unimodular? (see later).

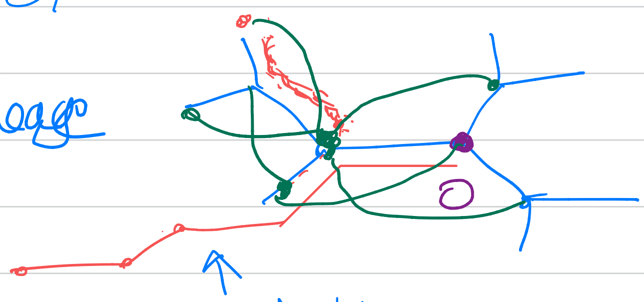
→ Reg. trees are conn. & symmetric.



– Peterson graph ✓ Smallest non-Cayley graph.

Eg: (Grandfather graph) G.

we can get ancestral line / lineage for all vertices



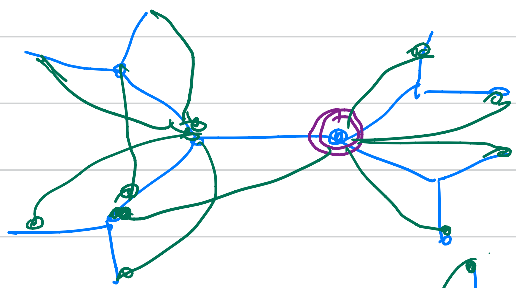
ancestral line of o. any unbdd path from o.

⇒ Every vertex has a lineage & in particular a grandfather.

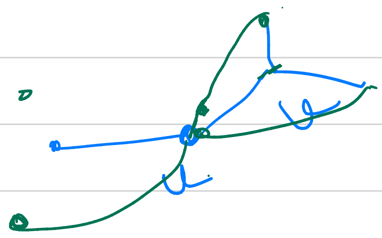
Connect each vertex to its grandfather.

G is transitive. (check).

G isn't symmetric.



$$(G,u,v) \not\cong (G,v,u).$$



$f(G,u,v) = \mathbb{1}[u$ is grandfather of $v]$

$$\sum_v f(G,u,v) = 4 \neq \sum_v f(G,v,u) = 1.$$

$\sum_v f(G,u,v)$ = # of grand-children of u; $\sum_v f(G,v,u)$ = # of grand-fathers of u.

$\Rightarrow (G,u)$ isn't unimodular for any $u \in V$.

Defn: $\mathcal{G}^f = \{ G : |G| < \infty \}$ – finite graphs.

$\mathcal{P}_s$ = "closure of $\mathcal{G}^f$ under LWC"

$= \{ (G,o) : G_n \xrightarrow{LW-d} (G,o)$ & $|G_n| < \infty \} \subseteq \mathcal{P}_*$

Clarification: $\mathcal{P}_*$ — prob. measures on $\mathcal{G}_*$ i.e., rooted random graphs.

$\mathcal{P}_s$ — Prob. measures on $\mathcal{G}_*$ obtained as a limit of unif. rooted finite graphs.

We have slightly abused notation by representing prob. measures by random elements.

A rooted (random) graph (G, o) is SOFIC if $(G, o) \in \bar{\mathcal{G}}^f$.

Qn: If $\mathcal{P}_s = \mathcal{P}_*$?

THM: Any Sofic graph is unimodular.
i.e., Unimodular random graphs are closed under LWC.

$\Rightarrow \mathcal{P}_s \overset{?}{\subseteq} \mathcal{P}_u \subsetneq \mathcal{P}_*$.

OPEN QUESTIONS — If $\mathcal{P}_s = \mathcal{P}_u$?

→ EXTRAS.

Suppose P is a group property.
Then we can say a graph G has property P
if Aut(G) has property P.
Gives a way to translate group-theoretic prop. to Graph properties.

Ref: Group invariant percoln on graphs
— Benjamini, Lyons, Peres, Schramm.
Processes on unimodular random networks
— Aldous & Steele.

Application of THM: Amenable Cayley graphs with arbitrary root are Sofic.

Ref for Unimodularity: Blaszczyszyn Ch 5 & Bordenave - Lecture.....