Recall: $\{G_n\}_{n\geq 1}$ - Sequence of dense graphs | Graphon: $k:[0,1]^2 \to [0,1]$
$E(G_n) \equiv$ order n^2 Symmetric m'ble

- Stated [LS 06] on deterministic limit theory
  - Equivalent metrics | Counts of subgraph.
- Proved LLN results for $G(n,k)$.
  - C.L.T.
  - Several generalisations
  - Testing

[Agenda 7 weeks] • Exchangeability Theory w.r.t graph limits

Section 1: Exchangeability / Exchangeable Arrays

$\{X_i\}_{i\geq 1}$ be a sequence of binary random variables.

They are exchangeable if

$$\mathbb{P}(X_1 = e_1, \dots, X_n = e_n) = \mathbb{P}(X_1 = e_{\sigma(1)}, \dots, X_n = e_{\sigma(n)})$$

for all n, permutation $\sigma \in \Pi_n$ and $e_i \in \{0,1\}$.

Theorem 1 [de Finetti] If $\{X_i\}_{i\geq 1}$ is a binary exchangeable sequence, then:

① $X_\infty = \lim\limits_{n\to\infty} \dfrac{X_1 + \dots + X_n}{n}$ exists w.p. 1

② If $\mu(A) = \mathbb{P}(X_\infty \in A)$, then $\forall n \geq 1$, e_i, $1 \leq i \leq n$

$$\mathbb{P}(X_1=e_1, \dots, X_n=e_n) = \int_0^1 x^s (1-x)^{n-s} \mu(dx)$$

$s = e_1+e_2+\dots+e_n$

Extensions:
- $X_i \in E$ — complete separable metric space
- partial exchangeability [kallenberg]

For graph limits:

Definition let $\{X_{ij}\}_{1\le i,j<\infty}$ be binary random variables

They are separately exchangeable if

$$\circledast \quad \mathbb{P}(X_{ij} = e_{ij}, 1\le i,j\le n) = \mathbb{P}(X_{ij} = e_{\sigma(i)\tau(j)}, 1\le i,j\le n)$$

for all n, all permutations $\sigma, \tau \in \Pi_n$ and $e_{ij} \in \{0,1\}$.

They are jointly exchangeable if $\circledast$ holds in the special case $\sigma=\tau$.

In other words: $\{X_{ij}\} \overset{d}{=} \{X_{\sigma(i)\tau(j)}\}$ — separately exchangeable

$\{X_{ij}\} \overset{d}{=} \{X_{\sigma(i)\sigma(j)}\}$ — jointly

all permutations $\sigma, \tau : \mathbb{N} \to \mathbb{N}$.

— let U_i, V_j $1\le i,j<\infty$ be uniform (independent) on $[0,1]$.

$W(\cdot,\cdot) : [0,1]^2 \to [0,1]$

$$\tilde{X}_{ij} = \begin{cases} 1 & \text{w.p. } W(U_i,V_j) \\ 0 & \text{w.p } 1-W(U_i,V_j) \end{cases}$$

$\mathbb{P}_w(\cdot)$ is the distribution $\{\tilde{X}_{ij}\}$. [Separately exchangeable by construction]

Theorem (Aldous - Hoover) :- Let $X = \{X_{ij}\}_{1 \le i,j < \infty}$ be a separately exchangeable binary array. Then there is a probability μ such that

$$\mathbb{P}(X \in A) = \int \mathbb{P}_w(A)\, \mu(dw)$$

Remarks :-

- a similar result exists for jointly exchangeable
- X_{ij} - can take values in a complete separable metric space
- uniqueness of μ. $\equiv$ discussion / results

Graph limits - Connections :

$w : [0,1]^2 \to [0,1)$ symmetric & measurable

$\{U_i\}_{i \ge 1}$ - sequence of independent uniform random variables

$V_n = \{1, \ldots, n\}$ connect $i \sim j$ w.p. $w(U_i, U_j)$

$$X_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

Reference :- Tim Austin - Notes on exchangeability

Definitions: G - simple graph - , $V(G) \equiv$ vertices $v(G) = |V(G)|$

[labelled / unlabelled] $E(G) \equiv$ Edges $e(G) = |E(G)|$

labels: $\{1, \dots, n\} := [n]$

$\mathcal{L}_n = 2^{\binom{n}{2}}$ labelled graphs on $[n]$.

$$\mathcal{L} := \bigcup_{n=1}^{\infty} \mathcal{L}_n$$

Unlabelled graph : is a labelled graph where we ignore the labels

$\mathcal{U}_n$ = set of unlabelled graphs of order n.
$\mathcal{L}_n / \cong$ of labelled graphs modulo isomorphism

$$\mathcal{U} := \bigcup_{n=1}^{\infty} \mathcal{U}_n$$

If G is an (unlabelled) graph & $v_1, v_2, \dots v_k$ is a sequence of vertices in G, then $G[v_1, v_2, \dots v_k]$ denotes the labelled graph with vertex set $[k]$.
[$v_i = v_j$ is allowed $\Rightarrow$ no edge between i & j]

$G[k]$: for $k \geq 1$ — random graph $G(v_1, \dots, v_k)$ obtained by Sampling $v_1, v_2, \dots v_k$ uniformly at random among vertices of G, with replacement. [$\{v_1, \dots, v_k\}$ are independent uniformly distributed vertices in G]

$G[k]'$ - $k \leq v(G)$, be the random graph $G(v_1', \dots, v_k')$ when we sample $v_1', \dots, v_k'$ uniformly at random without replacement.

[$\{v_1, \dots, v_k\}$ are independent uniformly distributed distinct vertices in G]

let F and G be two graphs.

$s(F,G) \equiv$ proportion of mappings $V(F) \to V(G)$ that are graph homomorphisms from $F \to G$.

i.e. $s(F,G)$ is the probability that a uniform random mapping $V(F) \to V(G)$ is a graph homomorphism.

i.e. $s(F,G) = \mathbb{P}(F \subseteq G[k])$ where $k = v(F)$.

- Both F and $G[k]$ are graphs on $[k]$.

$F \subseteq G[k]$ $\quad V(F) = V(G[k]) = [k]$ $\in E(F) \subseteq E(G[k])$

Ex: $t(F,G) = \mathbb{P}(F \subseteq G[k]')$

Convergence & Topology:

$\{G_n\}$ - sequence of graph converges if $s(F,G_n)$ converges for every graph F.

$\tau : \mathcal{U} \longrightarrow [0,1]^{\mathcal{U}}$ defined by

$$\tau(G) := [\, s(F,G) \,]_{F \in \mathcal{U}} \in [0,1]^{\mathcal{U}}$$

$\{G_n\}_{n \geq 1}$ converges $\iff$ $\{\tau(G_n)\}$ converges in $[0,1]^{\mathcal{U}}$ equipped with product topology.

$[0,1]^{\mathcal{U}}$ - compact metric space

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \, \delta_i(x_i,y_i)$$

$\mathcal{U}^* := \tau(\mathcal{U}) \subseteq [0,1]^{\mathcal{U}}$ – image of $\mathcal{U}$ under τ

$\overline{\mathcal{U}^*}$ be the closure of $\mathcal{U}^*$ in $[0,1]^{\mathcal{U}}$

[$\overline{\mathcal{U}^*}$ – compact metric ; – [LSOL] – description of it] already seen

Remarks:- • τ – is not injective – Ex: Find example.

• $G \longleftrightarrow$ identified with $\tau(G)$

$\{G_n\}$ convergence — convergence in $\mathcal{U}^*$

• There are many metric on $\mathcal{U}^*$

G – identify everyone with same image in $\mathcal{U}^*$ – technical tedious

$\mathcal{U}^+ = \mathcal{U} \cup \text{"}\{*\}\text{"}$

$\tau^+ : \mathcal{U} \longrightarrow [0,1]^{\mathcal{U}} \times [0,1]$

$\tau^+(G) = \left(\tau(G), \frac{1}{v(G)}\right)$

Ex: τ^+ is injective

Can do: • $\mathcal{U} \longrightarrow \tau^+(\mathcal{U}) \in [0,1]^{\mathcal{U}^+}$

$\overline{\mathcal{U}} \subseteq [0,1]^{\mathcal{U}^+}$

$\{G_n\}_{n\geq 1}$ converges in $\overline{\mathcal{U}^*}$ iff

• $v(G_n) \to \infty$ & $\{G_n\}_{n\geq 1}$ converges in $\overline{\mathcal{U}^*}$

or

• $\{G_n\}_{n\geq 1}$ is eventually a constant sequence