16/11 : L11 : SOFICITY & UNIMODULARITY.

$$\mathcal{P}_f := \{ (G, o) : G \text{ finite}, \ o \overset{d}{=} \mathrm{Unif}([v]) \}$$

$\mathcal{P}_f \subseteq \mathcal{P}_u$;

<u>THM:</u> $\mathcal{P}_u$ is closed under local weak convergence.

$$\Rightarrow \mathcal{P}_s = \overline{\mathcal{P}_f} \subseteq \mathcal{P}_u .$$

<u>Proof:</u> let $(G_n, o_n) \in \mathcal{P}_u \ni (G_n, o_n) \xrightarrow{LW-d} (G, o)$.

let $f : \mathcal{G}_{**} \to \mathbb{R}_+$.

let $t > 0$ & $g \in \mathcal{G}_* \ni g \subseteq B_r^{(g)}(o)$.

$$f_{t,g}(G,u,v) = t \wedge f(G,u,v) \mathbb{1}[d_G(u,v) \le t] \times \mathbb{1}[(G,u)_t \cong g].$$

$$\Rightarrow \overline{f}_{t,g}(G,u) = \sum_{v \in V_n} f_{t,g}(G,u,v)$$

$$= \sum_{v \in B_t^G(u)} [t \wedge f(G,u,v)] \, \mathbb{1}[(G,u)_t \cong g]$$

$$\underline{f}_{t,g}(G,u) = \sum_{v \in V_n} f_{t,g}(G,v,u)$$

$$= \sum_{v \in B_t^G(u)} t \wedge f(G,v,u) \, \mathbb{1}[(G,v)_t \cong g]$$

$F_{t,g}$ is bdd & depends on t-nbhd of root.

$f_{t,g}$ is also bdd & depends on $2t$-nbhd of root. ??

$\Rightarrow$ $F_{t,g}$, $f_{t,g}$ are bdd cts fnls on $\mathcal{G}_*$.

$$\Rightarrow \quad \mathbb{E}[F_{t,g}(G_n,o_n)] \overset{\text{unimodularity of }(G_n,o_n)}{=} \mathbb{E}[f_{t,g}(G_n,o_n)]$$

(by LWC) $\longrightarrow$

$$\mathbb{E}[F_{t,g}(G,o)] = \mathbb{E}[f_{t,g}(G,o)]$$

$\Rightarrow$ (G,o) satisfies MTP for $f_{t,g}$.

Summing over g & Fubini-Tonelli,

(G,o) satisfies MTP for $f_t(G,u,v) = (t \wedge f(G,u,v)) \times \mathbb{1}[d_G(u,v) \le t]$

$$\Rightarrow \quad \mathbb{E}\Big[\sum_{v\in V} f_t(G,o,v)\Big] = \mathbb{E}\Big[\sum_{v\in V} f_t(G,v,o)\Big]$$

$\forall v \in V$, $f_t(G,u,v) \uparrow f(G,u,v)$ as $t \to \infty$

By MCT,

$$\mathbb{E}\Big[\sum_{v\in V} f_t(G,o,v)\Big] = \mathbb{E}\Big[\sum_{v\in V} f_t(G,v,o)\Big]$$

$$\downarrow t\to\infty \qquad\qquad \downarrow t\to\infty$$

$$\mathbb{E}\Big[\sum_{v\in V} f(G,o,v)\Big] = \mathbb{E}\Big[\sum_{v\in V} f(G,v,o)\Big]$$

∎

$\mathcal{P}_u(\mathcal{T})$ = Unimodular random graphs supported on rooted trees

$\mathcal{P}_u(\mathcal{T}) \subseteq \mathcal{P}_s$ Elek, Bowen

See Bordenave : Counting & optimizing ....

$\mathcal{P}_s = \mathcal{P}_u$?

Eg: (1) Amenable Cayley graphs are sofic.

$B_r^G(o) \xrightarrow{LW-C} (G, o)$ for amenable Cayley graphs

[Assignment 4].

(2) $BGW(\lambda)$ is sofic & so unimodular.

(3) (Ex.) G - Amenable Cayley graph.

$G_p = (V, E_p) \subseteq G$

$E_p = \{e \in E : X_e = 1\}$ - Bond percolation.

$X_e, e \in E$ i.i.d. $Ber(p)$.

Is (G_p, o) unimodular ?

(4) Cayley graphs: $f : \mathcal{G}_{**} \to \mathbb{R}_+$ G- cayley graph.

$\Rightarrow f(G, u, v) = f(G, \gamma u, \gamma v)$, $\gamma \in G$.

$\hookrightarrow$ invariant by left-multiplication.

as $(G, u, v) \cong (G, \gamma u, \gamma v)$, $\gamma \in G$.

$$\sum_{v\in G} f(G,0,v) = \sum_{v\in G} f(G,0,v^{-1}), \quad 0 - \text{identity}.$$

(left multiply by v) $$= \sum_{v\in G} f(G,v,0)$$

$\Rightarrow$ $(G,0)$ is unimodular.

THM: (Induction invariance):

Let $(G,0) \in \mathcal{G}^* \ni$ MTP holds for all $f : \mathcal{G}_{**} \to \mathbb{R}_+ \ni f(G,u,v) = 0$ if $u \not\sim v$.

The $(G,0)$ is unimodular.

Proof: $f : \mathcal{G}_{**} \to \mathbb{R}_+$ can be written as

$$f(G,u,v) = \sum_{t=0}^{\infty} f_t(G,u,v)$$

where $f_t(G,u,v) = 0$ if $d(u,v) \neq t$.

Assumption is MTP for f_1.

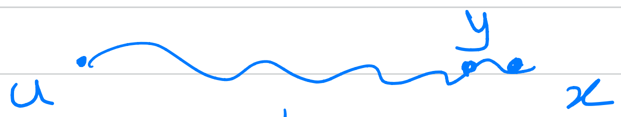
Set $f = f_t$ for some $t \geq 2$. We'll prove by induction.

$\forall\, k \geq 1 \quad \partial B_k^G(u) = B_k^G(u) \setminus B_{k-1}^G(u)$

If $x \in \partial B_t^G(u)$, let $\pi(G,u,x) = \#$ geodesic paths from u to x ≥ 1

geodesic = shortest paths = paths of length $d(u,x)$.

For $y \in \partial B_{t-1}^G(u)$, $\pi(G,u,x,y) = \#$ geodesic paths from u to x that hit y first.



$$\pi(G,u,x) = \sum_{y \in \partial B_{t-1}^G} \pi(G,u,x,y) \quad — \quad (1)$$

$\forall\, y \in \partial B_{t-1}^G(u)$, $$h(G,u,y) = \sum_{x \in \partial B_t^G(u)} f(G,u,x) \frac{\pi(G,u,x,y)}{\pi(G,u,x)}$$

& $h(G,u,y) = 0 \quad \forall\, y \notin \partial B^G_{t+1}(u)$.

$$\sum_{v\in V} h(G,u,v) = \sum_{y\in\partial B^G_{t+1}(u)} \sum_{x\in \partial B^G_t(u)} f(G,u,x)\, \frac{\pi(G,u,x,y)}{\pi(G,u,x)}$$

(interchange sums) $$= \sum_{x\in\partial B^G_t(u)} \sum_{y\in\partial B^G_{t+1}(u)} f(G,u,x)\, \frac{\pi(G,u,x,y)}{\pi(G,u,x)}$$

(by ①) $$= \sum_{x\in \partial B^G_t(u)} f(G,u,x)$$

$$= \sum_{v\in V} f(G,u,v).$$

By inductive assumption, h satisfies MTP & now so does f. ☒

We know BGW(λ) is unimodular.

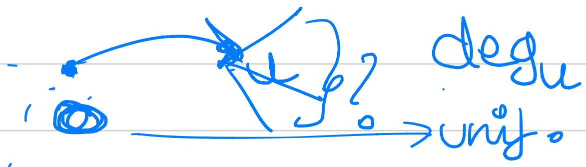
Ex. S.T. BGW(P) is unimodular iff P is Poisson(λ) distribn.

Let P be pmf on $\mathbb{Z}_+$, the offspring distribn & $\sum \ell P(\ell) < \infty$

Let $\hat{P}(k-1) = \dfrac{k P(k)}{\sum_{\ell\geq 0} \ell P(\ell)}$ be the size-biased distribn of P.

Ex. when is $\hat{P} = P$?

If degrees are i.i.d. P,

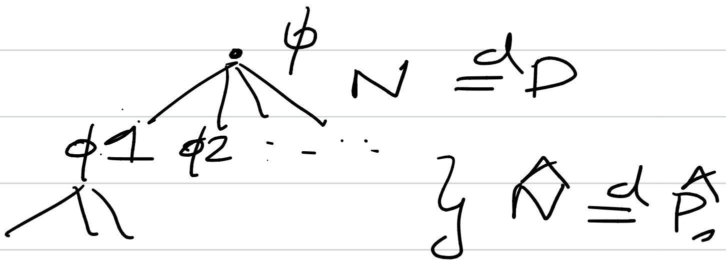


$\deg_u - 1 \overset{d}{=}$?

(equivalently, $\dfrac{\#\{u : \deg_u = k\}}{|V|} = P(k)$)

Defn: UGW(P) is the random tree $\ni$ $N_\phi \overset{d}{=} P$, $N_i \overset{d}{=} \hat{P}$ $\forall\, i \neq \phi$.

& N_ϕ, N_i are indep.



THM: UGW(P) is a unimodular random graph.

Eg: $P(k) = \delta_d(k) = \mathbb{1}[k=d]$, $d \geq 2$.

$\hat{P}(k-1) = \dfrac{k P(k)}{d} = \dfrac{d\, \mathbb{1}[k=d]}{d} = \mathbb{1}[k-1 = d-1]$.

UGW(P) is d-regular tree.

$\Rightarrow$ d-regular trees are unimodular.

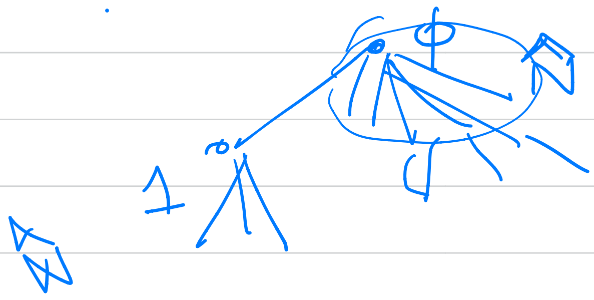
Eg: UGW(P) are loc'c limits of configuration models.

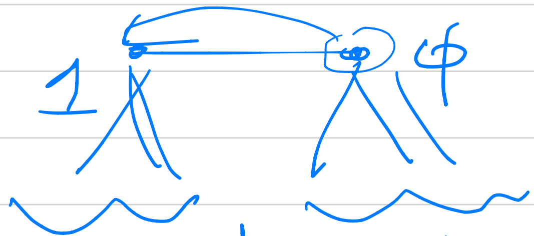
Proof:

ETPT MTP for $f \ni f(G,u,v) = 0$ if $u \nsim v$.

$$\mathbb{E}\left[\sum_{i=1}^{N_\phi} f(T,\phi,i)\right] \underset{(?)}{=} \sum_{k=1}^{\infty} k\,P(k)\,\mathbb{E}\left[f(T,\phi,1) \mid N_\phi = k\right]$$

$$= \underline{\mathbb{E}N} \sum_{k=0}^{\infty} \hat{P}(k)\,\mathbb{E}\left[f(T,\phi,1) ; N_\phi = k+1\right]$$





$T_1 \overset{d}{=} T' \overset{d}{=} GWT(\hat{P})$