Due: December 1, 2020

1. Let $\mathcal{U}$ be the set of all unlabelled graphs. Consider the map $\tau : \mathcal{U} \to [0,1]^{\mathcal{U}}$ defined by

$$\tau(G) := (s(F,G))_{F\in\mathcal{U}}.$$

   Show that

   (a) Show that $[0,1]^{\mathcal{U}}$ is a compact metric space.

   (b) Show that τ is not injective.

   (c) Show that $\tau^+ : \mathcal{U} \to [0,1]^{\mathcal{U}} \times [0,1]$ defined by

$$\tau^+(G) = (\tau(G), v(G)^{-1})$$

   is injective.

2. Let s, t, t_{ind} be the functions as discussed in class. Show that

   (a) Let $\mathcal{L}_k$ be the set of all labelled graph on $[k]$ vertices. Show that

$$\mid s(F,G) - t(F,G) \mid \leq \frac{v(F)^2}{2v(G)}$$

$$t(F,G) = \sum_{H\in\mathcal{L}_k, H\supset F} t_{\text{ind}}(F,G)$$

   and

$$t_{\text{ind}}(F,G) = \sum_{H\in\mathcal{L}_k, H\supset F} (-1)^{e(H)-e(F)} t(F,G).$$

3. Let G_n be a graph with vertex set $[n]$. Let $\hat{G_n}$ be the random labelled graph obtained by a random re-labelling of the vertices of G. Show that for any $F \in \mathcal{L}_k$,

$$\mathbb{E}(t_{\text{ind}}(F,G_n) \leq \mathbb{P}(\hat{G_n}|_{[k]} = F) \leq \mathbb{E}(t_{\text{ind}}(F,G_n) + P(v(G_n) \leq k)$$

4. Let H be an exchangeable random infinite graph in $\mathcal{L})\infty$. Show that the following are equivalent

   (a) H is exchangeable

   (b) $H|_{[k]}$ has a distribution invariant under all permutations of $[k]$ for all $k \geq 1$.