24/11/20 : L12 - DOUBLY ROOTED & WEIGHTED GRAPHS

Unimodularity of UGW(P), Please see Bordenave (Random graphs & Comb. optimization) Lemma 3.10.

Proof uses change of measure from P to $\hat{P}$ and symmetry under $\hat{P}$.

$\mathcal{G}_{**}$ - Doubly rooted graphs

$(G,u,v) \simeq (G',u',v')$ if $G \simeq G'$ with $u \mapsto u'$, $v \mapsto v'$.

$B_t^G(u;v) = B_t^G(u) \cup B_t^G(v)$

$=$ induced subgraph on $\{w : \min\{d_G(u,w), d_G(v,w)\} \le t\}$

We define metric on $\mathcal{G}_{**}$ similar to the local weak metric on $\mathcal{G}_*$.

$R(G,G') = \sup\{r \in \mathbb{N} : B_r^G(u,v) \simeq B_r^{G'}(u',v')\}$

$$d_*((G,u,v),(G',u',v')) = \frac{1}{R(G,G')+1}$$

Ex : S.T. $(\mathcal{G}_{**}, d_*)$ is a complete seperable metric space.
(A)

Ex : S.T. $\pi \quad (G,u,v) \longmapsto (G,u)$ is continuous.
(A)

THM: $\mathcal{P}_u$ is closed under local weak convergence.

$$\Rightarrow \mathcal{P}_s = \overline{\mathcal{P}_{ff}} \subseteq \mathcal{P}_u .$$

Proof: If MTP holds for $f_t : \mathcal{G}_{**} \to \mathbb{R}_+$ $\ni f_t(G,u,v) = 0$ iff $d_G(u,v) > t$ (with $t > 0$)

then MTP holds, $\forall f$. (arbitrary $f = \sum_{t=0}^{\infty} f_t \cdots$).

Fix $t > 0$ & $f \ni f(G,u,v) = 0$ if $d_G(u,v) > t$.

$\exists$ simple functions $f_n \ni f_n \uparrow f$ & $f_n = 0$ if $d_G(u,v) > t$.

Simple function f is of form

$$f([G,u,v]) = \sum_{l=1}^{L} \alpha_l \mathbb{1}[[G,u,v] \in A_l] \times \mathbb{1}[d_G(u,v) \le t], \quad A_l \in \mathcal{B}(\mathcal{G}_{**}), \quad \alpha_l \ge 0.$$

By linearity, it is enough to prove MTP

for $f([G,u,v]) = \mathbb{1}[[G,u,v] \in A] \mathbb{1}[d_G(u,v) \le t]$, $A \in \mathcal{B}(\mathcal{G}_{**})$

$\mathcal{A} = \{ A \in \mathcal{B}(\mathcal{G}_{**}) : f([G,u,v])$ as above satisfies MTP $\}$

$\mathcal{A}$ is closed under countable union (by MCT) & complements.

Also trivially true for $A = \mathcal{G}_{**}$.

So $\mathbb{1}[G \in A^c] \mathbb{1}[d_G(u,v) \le t] = \mathbb{1}[d_G(u,v) \le t] - \mathbb{1}[G \in A] \mathbb{1}[d_G(u,v) \le t]$

So enough to prove MTP for A in a generating class. $\mathcal{A}_*$.

Define $\mathcal{A}_* = \{ A_{s,(H,i,j)} : s > 0, (H,i,j)$ is a finite
$(H,i,j) \cong B^H_s(i,j)$. $\leftarrow$ s-depth graph $\ni d_H(i,j) \le t \}$

$$A_{s,(H,i,j)} := \{ (G,u,v) : B^G_s(u,v) \cong (H,i,j) \}$$

Since $\sigma(\mathcal{A}_*) \overset{\text{(check)}}{=} \mathcal{B}(\mathcal{G}_{**})$ & so ETPT $\mathcal{A}_* \subseteq \mathcal{A}$.

Consider $A_{s,(H,i,j)}$

Set
$$\underline{h}((G,u)) = \sum_{v \in B^G_t(u)} \mathbb{1}[B^G_s(u,v) \cong (H,i,j)]$$
$$= \sum_{v \in V} \mathbb{1}[B^G_s(u,v) \cong (H,i,j)] \times \mathbb{1}[d_G(u,v) \le t]$$
$$= \sum_{v \in V} \mathbb{1}[(G,u,v) \in A_{s,(H,i,j)}] \mathbb{1}[d_G(u,v) \le t]$$

$$\bar{h}((G,u)) = \sum_{v \in B^G_t(u)} \mathbb{1}[B^G_s(v,u) \cong (H,i,j)]$$

$\Rightarrow$ $\bar{h}$ & $\underline{h}$ are cts fns on $\mathcal{G}_*$ as they depend on $B^G_{s+t}(u)$.

$\Rightarrow$ Suppose $s > t$ & $\underline{h}((G,u)) > 0$.

Then $B_t^{G}(u) \cong B_t^{H}(i)$

$\Rightarrow \quad \underline{h}((G,u)) \le |B_t^{H}(i)|$

$|||^{ly} \quad \overline{h}((G,u)) \le |B_t^{H}(i)|$

Thus $\underline{h}$ & $\overline{h}$ are bdd cts fns on $\mathcal{G}_*$.

Now $(G_n, O_n) \xrightarrow{LW\text{-}d} (G,O)$.

So $\mathbb{E}\,\underline{h}(G_n,O_n) \longrightarrow \mathbb{E}\,\underline{h}(G,O)$.

& $\mathbb{E}\,\overline{h}(G_n,O_n) \longrightarrow \mathbb{E}\,\overline{h}(G,O)$.

Ex: Prove via Lusin's thm.

By assumption (G_n,O_n) is unimodular & so

$$\mathbb{E}\,\underline{h}(G_n,O_n) = \mathbb{E}\,\overline{h}(G_n,O_n)$$

$$\Rightarrow \mathbb{E}\,\underline{h}(G,O) = \mathbb{E}\,\overline{h}(G,O)$$

$\Rightarrow \quad \mathcal{A}_* \subseteq \mathcal{A}$ as required. ∎

$\mathcal{G}_E$ - Rooted graphs with edge weights.

A $\mathbb{R}$-weighted graph (G,w) is a graph $G=(V,E)$ with weight fn $w: V^2 \to \mathbb{R}$ $\ni$ $w(u,v)=0$ if $(u,v) \notin E$.

w is EDGE-SYMMETRIC if $w(u,v) = w(v,u)$ & $w(u,u)=0$.

(G,w) is LOCALLY FINITE if $\forall\, v \in V$

$$\sum_{u \in V} \big(|w(u,v)| \vee |w(v,u)|\big)\mathbb{1}[(u,v)\in E] < \infty.$$

G loc. fin $\Rightarrow$ (G,w) is loc. fin.

A network is (G,ω) where $G=(V,E)$ is a lo. fin. graph & $\omega : (V \cup E) \to \Omega$, (Polish space).

Mark space

$\boxed{\text{DEFN}}$ $(G,\omega) \cong (G',\omega')$ if $\exists$ a graph isomorphism $\phi : G \to G'$ $\ni$ $\omega'(\phi(v)) = \omega(v)$, $\omega'(\phi(e)) = \omega(e)$ $\forall$ $v \in V$ & $e \in E$. [Network isomorphism]

$(G,o,\omega) \cong (G',o',\omega')$ if $\phi : (G,\omega) \to (G',\omega')$ isomorphism s.t. $\phi(o) = o'$. [Rooted network isomorphism]

$\underline{\mathcal{G}_*(\Omega)} := \{ \underbrace{[G,o,\omega]}_{\text{eq. class}} : (G,o,\omega) \text{ rooted network} \}$

$g_1, g_2 \in \mathcal{G}_*(\Omega)$ $\qquad g_i = (G_i, o_i, \omega_i)$

$$d(g_1,g_2) = \frac{1}{1+T}$$

where $T = \sup \{ t>0 : \exists$ rooted isomorphism $\phi : B_t^{g_1}(o_1) \to B_t^{g_2}(o_2) \ni d_\Omega(\omega_1(v), \omega_2(\phi(v))) \le 1/t$ & $d_\Omega(\omega_1(e), \omega_2(\phi(e))) \le 1/t$ $\forall v, e \in B_t^{g_1}(o_1) \}$.

Ex : $\mathcal{G}_*(\Omega)$ is a seperable & complete m. space.
(A)

Ex : Let $\psi : \mathcal{G}_* \to \Omega$, $\phi : \mathcal{G}_{**} \to \Omega$ m'ble.
(A)
Define (G,ω) as $\omega(u) = \psi(G,u)$
$\omega(u,v) = \phi(G,u,v)$.

If (G,o) is unimodular, so is (G,o,ω).

$\mathcal{G}_{**}(\Omega) = \{[G,u,v,\omega] : (G,u,v,\omega)$ - doubly rooted network$\}$

$d(g_1,g_2) = \frac{1}{1+T}$ $\qquad g_i = (g_i, u_i, v_i, w_i)$

$T = \sup\{t>0 : B_t^{g_1}(u_1,v_1) \overset{\phi}{\simeq} B_t^{g_2}(u_2,v_2) \ni$
$d_\Omega(\omega_1(v), \omega_2(\phi(v))) \le 1/t, \quad \forall\, v \in B_t^{g_1}(u_1,v_1)$
$d_\Omega(\omega_1(e), \omega_2(\phi(e))) \le 1/t \;\forall\, e \in B_t^{g_1}(u_1,v_1)\}$

(G,o,ω) is unimodular if $\forall f : \mathcal{G}_{**}(\Omega) \to \mathbb{R}_+$

$$\mathbb{E}\Big[\sum_{v\in V} f(G,o,v,\omega)\Big] = \mathbb{E}\Big[\sum_{v\in V} f(G,v,o,\omega)\Big].$$

Ex (A) Percolation preserves unimodularity.

Let (G,o,ω) be a unimodular random network.
Let $B \in \mathcal{B}(\Omega)$

Define $\hat{G} = (V(G), \hat{E} = \{(u,v) \in E : \omega(u,v) \in B, \omega(v,u) \in B\})$

$\hat{\omega}(u,v) = \omega(u,v)\, \mathbb{1}[\omega(u,v)\in B]\, \mathbb{1}[\omega(v,u)\in B]$.

S.T. $(\hat{G},o)$ is unimodular. What about $(\hat{G},o,\hat{\omega})$?

EXAMPLES OF RANDOM GRAPH MODELS.

(1) $V = \{0,1\}^n$ $\qquad v_i \sim v_j$ if $v_i - v_j = \pm e_k \quad 1 \le k \le n$
$\qquad e_k = (0\cdots, \underset{k}{1}, \cdots 0)$

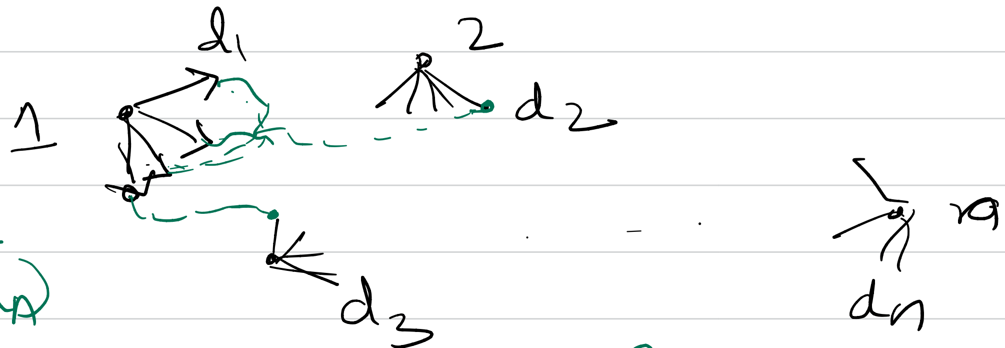
$p = c/n$. $\quad H(n,p) =$ Each edge is preserved w.p. p & indep.

$\deg(o \cdots o) \overset{d}{=} \mathrm{Bin}(n, p) \xrightarrow{d} \mathrm{Poi}(c)$.

$H(n,p) \xrightarrow{LW-P} \mathrm{BGW}(\mathrm{Poi}(c))$??

2) Configuration Model (vd Hofstad, Ch 3
Blaszczyszyn, Ch 4)

Let $\bar{d}_n = (d_{i,n})_{i=1}^{n}$ be a plausible degree sequence.
[Erdős – Gallai theorem]



Pair half-edges uniformly at random.

$G(n, \bar{d}_n)$

Repeat until you get a simple graph.

$$\deg(o) \overset{d}{=} \frac{1}{n} \sum_{i=1}^{n} \delta_{d_{i,n}}$$

$$\mathbb{P}(\deg(o) = k) = \frac{\#\{i : d_{i,n} = k\}}{n} = p_{k,n}.$$

Suppose $p_{k,n} \xrightarrow{n} p_k \ \forall k \geq 0$ & $\sum_{k \geq 0}^{\infty} p_k = 1$.

Does $G(n, \bar{d}_n) \xrightarrow{LW-C} \mathrm{BGW}(P)$? $P = (p_k)_{k \geq 0}$

But $\mathrm{BGW}(P)$ is not unimodular unless $P \overset{d}{=} \mathrm{Poi}(\lambda)$.

But what is $P \neq \mathrm{Poi}(\lambda)$?

By unimod. considerations, $G(n, \bar{d}_n) \xrightarrow{LW-C} \mathrm{UGW}(P)$.