1/12/2020 : L13 – Aldous-Steele Continuity Theorem

Let $\bar{G} = (V, E)$ be a finite connected graph, $\omega : V^2 \to \mathbb{R}_+$ an edge-symmetric weight function and $G = (\bar{G}, w)$ the associated weighted graph. We assume that for any subsets $A \neq B$ of E,

$$\sum_{e \in A} \omega(e) \neq \sum_{e \in B} \omega(e). \qquad (6)$$

The *minimal spanning tree* $\mathrm{MST}(G)$ is the unique minimizer of

$$\tau(G) = \min_{T \in \mathrm{ST}(G)} \sum_{e \in E(T)} \omega(e), \qquad (7)$$

where $\mathrm{ST}(G)$ is the set of spanning trees of $\bar{G}$ (uniqueness is a consequence of (6)). For $t > 0$, let $G(t) = (V, E(t))$, where $E(t) = \{e \in E : \omega(e) < t\}$ and for $v \in V$, let $G(t, v)$ be the connected component of v in $G(t)$. The following standard lemma gives a criterion to build to the minimal spanning tree.

Lemma 3.1. *An edge $e = \{u, v\} \in E$ belongs to $\mathrm{MST}(G)$ if and only if $G(\omega(e), u) \cap G(\omega(e), v) = \emptyset$.*

Lemma 3.1 can be used as a criterion to define the minimal spanning forest of an infinite graph.

Definition 3.2 (Minimal spanning forest). *Let $G = (\bar{G}, \omega)$ be a locally finite weighted graph with $\omega : V^2 \to \mathbb{R}_+$ edge-symmetric such that (6) holds for all finite subsets $A \neq B$ of edges in E. The minimal spanning forest of G, $\mathrm{MSF}(G)$ is the graph with vertex set V and edges, the set of $e \in E$ such that (i) $G(\omega(e), u) \cap G(\omega(e), v) = \emptyset$ and (ii) $G(\omega(e), u)$ and $G(\omega(e), v)$ are not both infinite.*

It is easy to check that $\mathrm{MSF}(G)$ is indeed a forest (no cycles). Also, by Lemma 3.1, if G is finite, our definition is consistent and $\mathrm{MSF}(G) = \mathrm{MST}(G)$. Finally, if G is an infinite graph, then each connected component of $\mathrm{MSF}(G)$ is infinite.

Aldous-Steele Continuity Theorem:

Let G_n be a connected finite graph with edge-symmetric weight $w_n \in \mathbb{R}_+$ & satisfying (6) for all $n \geq 1$. Let $(G_n, w_n) \xrightarrow{\text{lw-d}} (G, w, o)$, a random rooted & weighted graph. Assume that (G, w, o) satisfies (6) a.s. Define (G_n, w_n') by

$$w_n'(e) = (w_n(e), \mathbb{1}[e \in \mathrm{MST}(G_n)])$$

& (G, w') by $w'(e) = (w(e), \mathbb{1}[e \in MSF(G)])$.

Then $(G_n, w_n') \xrightarrow{LW-d} (G, w', o)$.

COROLLARY: Same assumptions as in the theorem.

Set $\ell_n(u) = \sum_{v \in V_n} w_n(u,v) \quad \forall\ u \in V_n$.

Assume that $\ell_n(O_n)$ is U.I. where $O_n \overset{d}{=} \mathrm{Unif}(V_n)$.

$$\frac{T(G_n)}{|V_n|} \longrightarrow \frac{1}{2} \mathbb{E}\Big[\sum_{v \in V} w(o,v) \mathbb{1}[(o,v) \in MSF(G)]\Big]$$

So holds if $(G_n)_{n \geq 1}$ has bounded degrees & weights.

(i.e., $\frac{T(G)}{|V|}$ is estimable on bounded degree & weights graphs).

Proof: Given $(\bar{G}, \bar{o}, \bar{w}') \in \mathcal{G}_*(\mathbb{R}_+ \times \{0,1\})$, with $\Omega = \mathbb{R}_+ \times \{0,1\}$

$$L(\bar{G}, \bar{o}) = \frac{1}{2} \sum_{v \in V} \bar{w}(o,v) \mathbb{1}[(o,v) \in MSF(\bar{G})] \leq \frac{\ell_n(o)}{2}$$

$$\mathbb{E}\, L(G_n, O_n) = \frac{T(G_n)}{|V_n|} = \frac{1}{|V_n|} \sum_{u \in V_n} L(G_n, u_n)$$

L is a continuous fn on $\mathcal{G}_*(\mathbb{R}_+ \times \{0,1\})$

As cty thm

$$\Rightarrow \mathbb{E}[\underbrace{L(G_n, O_n) \wedge t}_{\text{cts \& bdd}}] \to \mathbb{E}[L(G,o) \wedge t] \quad \forall\ t > 0. \quad (1)$$

$$\mathbb{E}[L(G_n, O_n)] = \mathbb{E}[L(G_n, O_n) \wedge t] + \mathbb{E}[(L(G_n, O_n) - t)\, \mathbb{1}[L(G_n, O_n) \geq t]]$$

$$\mathbb{E}[(L(G_n, O_n) - t) \mathbb{1}[L(G_n, O_n) \geq t]] \leq \mathbb{E}[L(G_n, O_n) \mathbb{1}[\quad \geq t]]$$

$$\leq \frac{1}{2} \mathbb{E}[\ell_n(O_n) \mathbb{1}[\ell_n(O_n) \geq 2t]] \leq \varepsilon \quad \forall\ n \geq 1 \ \& \ t \text{ large.}$$

$\hookrightarrow$ U.I. of $\ell_n(O_n)$

Choose t large as above

$$E[L(G_n,O_n)\wedge t] \le E[L(G_n,O_n)] \le E[L(G_n,O_n)\wedge t]+\varepsilon$$

(1) $\Rightarrow$ $$E[L(G,O)\wedge t] \le \text{“}\lim_{n\to\infty} \frac{T(G_n)}{|V_n|}\text{”} \le E[L(G,O)\wedge t]+\varepsilon$$

By MCT, $E[L(G,O)\wedge t] \uparrow E[L(G,O)]$ as $t\uparrow\infty$.

$\Rightarrow$ $$E[L(G,O)] \le \text{“}\lim_{n\to\infty} \frac{T(G_n)}{|V_n|}\text{”} \le E[L(G,O)]+\varepsilon.$$

∎

<u>THM:</u> Let (G,O) be an a.s. ∞ unimodular random graph. Then $E\deg_G(O)\ge 2$.

<u>Proof:</u> $e=(u,v)\in E$.

$l(u,v) = (f,f),\ (f,\infty),\ (\infty,f),\ (\infty,\infty)$

depending on whether component of u & v in $G\setminus\{e\}$ is finite or ∞.

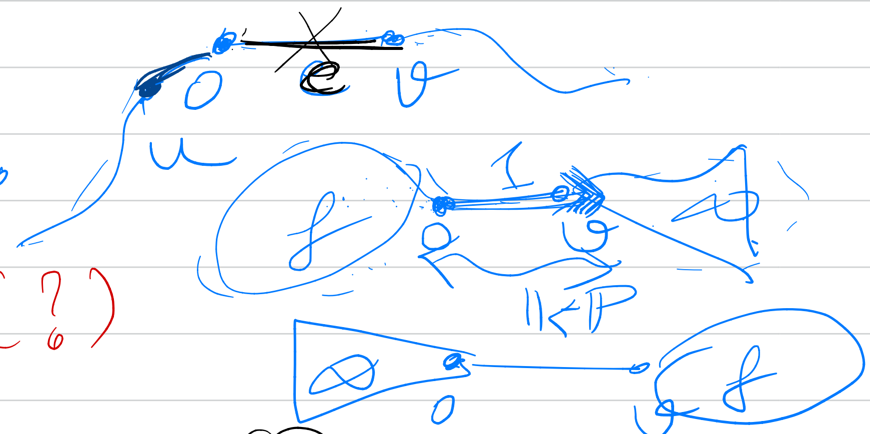
Let $a,b\in\{f,\infty\}$. $D_{a,b}$ = # of edges (O,v) ∋ $l(O,v)=(a,b)$.

a.s. $D_{f,f}=0$ as (G,O) is ∞ a.s.

If $l(O,v)=(\infty,\infty)$ then $\exists\, u$ ∋ $(O,u)\in E$ & $l(O,u)=(\infty,\infty)$

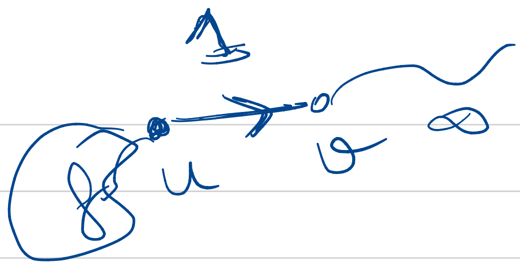
$\Rightarrow$ if $D_{\infty,\infty}\ge 1$ then $D_{\infty,\infty}\ge 2$.

If $D_{\infty,\infty}=0$ then $D_{f,\infty}\ge 1$ (?)



$\Rightarrow$ $2D_{f,\infty}+D_{\infty,\infty}\ge 2$ — (1)

$$f(G,u,v) = \mathbb{1}[\ell(u,v) = (f,\infty)]$$



By unimodularity,

$$\mathbb{E}\Big[\sum_{v\in V} f(G,0,v)\Big] = \mathbb{E}[D_{f,\infty}]$$

$$\|$$

$$\mathbb{E}\Big[\sum_{v\in V} f(G,v,0)\Big] = \mathbb{E}[D_{\infty,f}] \quad - \ ②$$

①+②

$$\Rightarrow \mathbb{E}[D_{\infty,f} + D_{f,\infty} + D_{\infty,\infty}] = \mathbb{E}[2D_{f,\infty} + D_{\infty,\infty}] \geq 2\varepsilon$$

$$\|$$

$(D_{ff} = 0$ a.s.$)$ $\leftarrow$ $\mathbb{E}[\deg_G(0)]$

$\boxtimes$

Proof of AS claim:

$$(G_n, w_n) \xrightarrow{d} (G, w).$$

(G_n, w_n') converges along a subsequence

as $w_n' \in \mathbb{R}_+ \times \{0,1\}$ & so (G_n, w_n') is tight.

WLOG assume $(G_n, w_n') \xrightarrow{LW\text{-}d} (G, \tilde{w}, o)$

where $\tilde{w}(e) = (w(e), \mathbb{1}[e \in S])$

for some $S \subseteq E(G)$.

[Instead of assuming $(G_{n_k}, w'_{n_k}) \xrightarrow{LW\text{-}d} (G, \tilde{w}, o)$ we've assumed that the seq. converges for convenience]

If we show $\tilde{w} = w'$ ($\Longleftrightarrow S = MSF(G)$ a.s.)

then we get uniqueness of the limit & thm follows.

Step 1: $MSF(G) \subseteq S$.

we know $(G_n, w_n', o_n) \xrightarrow{d} (G, \tilde{w}, o)$

By Skorohod's representation thm, $\exists$ a prob. space $\ni$ $d_*((G_n, w_n, o_n), (G, \tilde{w}, o)) \xrightarrow{n\to\infty} 0$ a.s. — $(*1)$

d_* – local weak metric on $\mathcal{G}_*(\Omega)$, $\Omega = \mathbb{R}_+ \times \{0,1\}$.

Let $e = (u,v) \in MSF(G) \Rightarrow G(w(e), u) \cap G(w(e), v) = \phi$ & one of them is finite.

WLOG assume $G(w(e), u)$ is finite.

$\exists$ t large $\ni$ $G(\underline{w(e)}, u) \subseteq B_t^G(o)$ [t is random dep. on (G,o)]

Because of uniqueness condn (6), we can choose $t \ni$

$$G(w(e) + \tfrac{1}{t}, u) = G(w(e), u)$$

From $(*1)$, for any t, $\exists$ n_0 (random) $\ni$

$\forall$ $n \geq n_0$, $B_{2t}^{G_n}(o_n) \overset{\phi}{\cong} B_{2t}^G(o)$

& $|w_n(e) - w(\phi(e))| \leq \frac{1}{2t}$; $\left(d_*((G_n, o_n, w_n), (G, o, w)) \leq \frac{1}{2t} \right)$

Let e_n be $\ni$ $\phi(e_n) = e$. $\Rightarrow |w(e_n) - w(e)| \leq \frac{1}{2t}$.

$e_n = (u_n, v_n)$, $\phi(u_n) = u$, $\phi(v_n) = v$.

$\Rightarrow$ $\exists$ no path from u to v in $G(w(e))$ i.e., $v \notin G(w(e), u) = G(w(e) + \frac{1}{t}, u) \subseteq B_t^G(o)$

Suppose $G(w(e_n), u_n) \not\subseteq B_t^{G_n}(o_n) \cong B_t^G(o)$

$\Rightarrow \exists$ $e_n' \in B_{t+1}(o_n) \setminus B_t(o_n)$ & $w(e_n') \leq w(e_n) \Rightarrow$

$w(\phi(e_n')) \leq w(e_n') + \frac{1}{2t} < w(e_n) + \frac{1}{2t} \leq w(e) + \frac{1}{t}$ — $(*2)$

$\Rightarrow$ $\phi(e_n') \notin B_t^G(o)$ & $\phi(e_n') \in G(w(e) + \frac{1}{t}, u)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Vert$

a contradiction. $\leftarrow$ $G(w(e), u) \subseteq B_t^G(o)$

$$\Rightarrow \quad G(w(e_n), u_n) \subseteq B_t^{G_n}(o_n)$$

$$\qquad \cong_\phi \qquad\qquad\qquad \cong_\phi$$

$$H_n \quad \subseteq \quad B_t^{G}(o)$$

Same argument as in $(*2)$ $\Rightarrow H_n \subseteq G(w(e) + \frac{1}{t}, u)$.

$$\Rightarrow \quad v \notin H_n \Rightarrow v_n = \phi^{-1}(v) \notin G(w(e_n), u_n)$$

$$\Rightarrow \quad e_n = (u_n, v_n) \in MST(G_n)$$

i.e., $e_n \in MST(G_n) \quad \forall \; n \geq n_0$

& $MST(G_n) \longrightarrow S$

$$\Rightarrow \quad e \in S.$$

Step 2 T.S.T. $MSF(G) \supseteq S$ a.s.

$$0 \leq \mathbb{E}\Big[\sum_{v \in V} 1\!\!1\, [\{o,v\} \in S \setminus MSF(G)]\Big]$$

$$= \mathbb{E}\deg_S(o) - \mathbb{E}\deg_{MSF(G)}(o)$$

$$\leq 2 - 2 = 0. \quad — \; (*3)$$

Fatou's lemma

$$\mathbb{E}\deg_S(o) = \mathbb{E}\lim_{n\to\infty} \deg_{MST(G_n)}(o)$$

$$\leq \liminf \mathbb{E}\deg_{MST(G_n)}(o)$$

MSF(G) has all comp. ∞ by prev lemma

$$\mathbb{E}\deg_{MSF(G)}(o) \geq 2$$

$$\mathbb{E}\deg_{MST(G_n)}(o) \leq \frac{1}{|V_n|}\sum_{v \in V_n} \deg_{MST(G_n)}(v) = \frac{2\,|\text{Edges of } MST_{G_n}(o)|}{|V_n|} \leq \frac{2(|V_n|-1)}{|V_n|} \leq 2$$

(*3) $\Rightarrow \mathbb{P}(\exists\, v \sim 0 \ni (0,v) \in S \setminus MSF(G)) = 0.$

Proof to be concluded by the next lemma. ∎

LEMMA: [Everything shows at the root]

Let $(G,0)$ be a unimodular randomly weighted rooted graph & $f: \mathcal{G}_* \to \{0,1\}$ be measurable.

If $f(G,0)=1$ a.s. then, $\forall v \in V$ $f(G,v)=1$ a.s.

Proof: $h(G,u,v) = \mathbb{1}[f(G,u) \neq 1]$

$$\mathbb{E}\Big[\sum_{v\in V} h(G,0,v)\Big] = \mathbb{E}\Big[\sum_{v\in V} \mathbb{1}[f(G,0)\neq 1]\Big] = 0$$

$\shortparallel$ MTP

$$\mathbb{E}\Big[\sum_{v\in V} h(G,v,0)\Big] = \mathbb{E}\Big[\sum_{v\in V} \mathbb{1}[f(G,v)\neq 1]\Big]$$

$$\Rightarrow \sum_{v\in V} \mathbb{1}[f(G,v)\neq 1] = 0 \quad \text{a.s.} \quad ∎$$

For prev. thm proof apply for

$f(G,0) = \mathbb{1}[\nexists \text{ no } (0,w) \in S \setminus MSF(G)].$