

Recall :-

$\mathcal{L}_\infty$ - labelled infinite graphs on $\mathbb{N}$
- compact metric space

$\mathcal{L}_n$ - graph labelled on $[n] \hookrightarrow \mathcal{L}_\infty$

$\hat{\mathcal{G}}$ - random labelled graph obtained by random relabelling of vertices of G by the numbers $\{1, \dots, v(G)\}$

$\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n = \mathcal{L} / \cong$ of labelled graphs modulo isomorphism

$\mathcal{U}_k = \bar{\mathcal{U}} / \mathcal{U}$

$s(F, G) := P(F \subseteq G[k])$

$t(F, G) = P(F \subseteq G[k])$

$t_{ind}(F, G) = P(F = G[k])$

$G \mapsto \tau(G) = (s(F, G))_{F \in \mathcal{U}} \hookrightarrow [0, 1]^{\mathcal{U}}$

$\tau(\mathcal{U}) = \mathcal{U}^* \in \bar{\mathcal{U}}^* \subseteq [0, 1]^{\mathcal{U}}$

$$|s(F, G) - t(F, G)| \leq \frac{v(F)^2}{2v(G)} \quad (*)$$

Ex: $t(F, G) = \sum_{F' \in \mathcal{L}_k, F' \supseteq F} t_{ind}(F', G) \quad (*')$

Ex' $t_{ind}(F, G) = \sum_{F' \in \mathcal{L}_k, F' \supseteq F} (-1)^{e(F') - e(F)} t(F', G) \quad (**)$

inclusion - exclusion

Theorem 4: $(G_n)_{n \geq 1}$ be a sequence of random graphs in $\mathcal{U} \in v(G_n) \xrightarrow{p} \infty$. TFAE.

(i) $G_n \xrightarrow{d} \Gamma$ in $\bar{\mathcal{U}}$ for some random $\Gamma \in \bar{\mathcal{U}}$

(ii) $G_n \xrightarrow{d} H$ in $\mathcal{L}_\infty$ for some random $H \in \mathcal{L}_\infty$

If the above hold then

$$P(H|_{[k]} = F) = E t_{ind}(F, \Gamma) \quad (*)$$

$$\forall F \in \mathcal{L}_k$$

Further $\Gamma \in \mathcal{U}_\infty$ a.s.

Definition: A random infinite graph $H \in \mathcal{L}_\infty$ is exchangeable if its distribution is invariant under every permutation of vertices

- equivalent to consider finite permutations

$$X_{ij} = 1_{((i,j) \in E(H))}$$

- jointly exchangeable

Corollary A: P is a random element in $\mathcal{U}_\infty = \bar{\mathcal{U}} \setminus \mathcal{U}$ & sequence $F_1, \dots, F_n$

$$E \prod_{i=1}^n s(F_i, P) = E s(\bigoplus_{i=1}^n F_i, P)$$

$$P \text{ distribution} \iff E(s(F, P))$$

$$\iff s, t, t_{ind}$$

Lemma : H be a random ^{infinite} graph in $\mathcal{L}_\infty$. T.F.A.E

- (i) H is exchangeable
- (ii) $H|_{[k]}$ has a distribution invariant under all permutations of $[k]$, for every $k \geq 1$
- (iii) $P(H|_{[k]} = F)$ depends only on the isomorphism type of F and can thus be seen as a function of F as an unlabelled graph in $\mathcal{U}_k$, $\forall k \geq 1$

Theorem 5: The limit H in Theorem 4 is exchangeable

Proof: By $\textcircled{A}$ H satisfies lemma (ii) above. $\Rightarrow H$ is exchangeable. $\square$

Theorem 6: - There a 1-1 correspondence between distributions of random elements of $\mathcal{U}_\infty$ (or $\bar{\mathcal{U}}^*$) and distribution of exchangeable graphs in $\mathcal{L}_\infty$

is given by

$P \in \mathcal{U}_\infty$, $\xleftrightarrow{1-1 \text{ correspondence}} H \in \mathcal{L}_\infty$

$$E(t_{\text{ind}}(F, P)) = P(H|_{[k]} = F) \quad \forall k \geq 1 \quad \textcircled{A}$$

or equivalently,

$$E(t(F, P)) = P(H \geq F) \quad \forall F \in \mathcal{L} \quad \textcircled{B}$$

Further $H|_{[n]} \xrightarrow{\mathcal{L}} P$ in $\bar{\mathcal{U}}$ as $n \rightarrow \infty$.

Proof: $(\neq)$ and $(\#)$ are equivalent because of

$$(\ast^0) \triangle (\ast^1) \in (\ast^0).$$

$$g(F, P) = t(F, P)$$

$$H \supset F \quad (\Rightarrow) \quad H|_{[k]} \geq F \quad \text{when } F \in \mathcal{L}_k.$$

To establish correspondence

$$P \in \mathcal{U}_\# \subseteq \bar{\mathcal{U}},$$

$\rightarrow u$ is dense in $\bar{\mathcal{U}}$

$\rightarrow H_n$ is a random unlabelled graph G_n

Such that $G_n \xrightarrow{a.s.} P$ in $\bar{\mathcal{U}}$ [Lemma A] $\Rightarrow$

$$V(G_n) \xrightarrow{p.s.} \in G_n \xrightarrow{d} P.$$

Theorem 4 & Theorem 5 $\exists H \in \mathcal{L}_\infty$, H exchangeable

$$\in \hat{\mathcal{G}}_n \xrightarrow{d} P$$

$\Rightarrow (\neq)$ is satisfied by $(\ast)$ in Theorem 4.

$(\neq)$ determines distributions of $H|_{[k]} \quad \forall k \geq 1.$

Conversely $H \in \mathcal{L}_\infty$ - exchange random graph

$$G_n = H|_{[n]}$$

• lemma (ii) $H|_{[n]}$ is invariant under permutations of $[n]$

$\Rightarrow G_n$ is invariant under permutations of $[n]$

$$\Rightarrow \hat{G}_n \stackrel{d}{=} G_n$$

($\mathcal{L}_\infty$ - product topology)

$$G_n \xrightarrow{d} H \text{ in } \mathcal{L}_\infty$$

$$\Rightarrow \hat{G}_n \xrightarrow{d} H \text{ in } \mathcal{L}_\infty$$

$$G_n \xrightarrow{d} \pi \text{ in } \mathcal{W}_\infty \Leftrightarrow$$

$(\oplus)$ holds

Theorem 4:

$(\oplus)$ determines distribution of π due to

Corollary A. $\square$

Probability Martingale facts

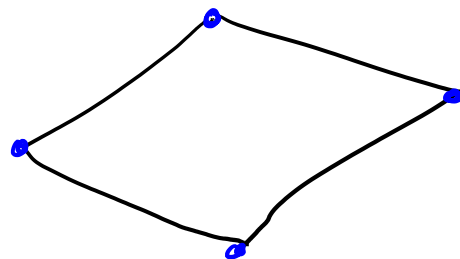
• Also $H|_{[n]} \xrightarrow{as} \pi$ because one can show

$$\left\{ t_{ne}(\mathbb{F}, H|_{[n]}) \right\}_{n \geq r(F)} \quad \forall F \in \pi$$

is a reverse martingale

• First law of large numbers proof : - we showed using concentration inequality $H|_{[n]} \xrightarrow{as} H$

- Extreme points of a convex set



Facts
need

- Space of measures — ? Extreme points

• tail σ -field — Kolmogorov 0-1 law