Recall:

- G – unlabelled graph
  $v_1, \dots, v_k$ are vertices in G
- $G(v_1, \dots, v_k)$ = labelled graph
  - $V = [k]$
  - $i \sim j$ if v_i, v_j are adjacent in G
    ($v_i = v_j$ no edge)
- $G[k]$ = random graph $G(v_1, \dots, v_k)$
  (with replacement) $v_i \sim$ uniform from $v(G)$
- (without replacement) $G[k]' \equiv$ random graph $G(v'_1, \dots, v'_k)$
  $v'_i \sim$ uniform from $v(G)$

F – fixed graph on k vertices

$$s(F, G) := \mathbb{P}(F \subseteq G[k])$$

Define:-

$$t(F,G) := \mathbb{P}(F \subseteq G[k]') \qquad k \leq v(G)$$

$$t_{ind}(F,G) := \mathbb{P}(F = G[k]')$$

$t_{ind}(F,G)$ & $t(F,G) \equiv 0$ if $k > v(G)$.

Ex:- $$|t(F,G) - s(F,G)| \leq k^2/2v(G)$$

Random Graph Convergence

Theorem: Let $\{G_n, n \geq 1\}$ be random unlabelled graphs & assume that $v(G_n) \xrightarrow{p} \infty$. Then T.F.A.E as $n \to \infty$

(i) $G_n \xrightarrow{d} \Gamma$ for some $\Gamma \in \overline{\mathcal{U}}$

(ii) $\forall$ finite family $F_1, \ldots, F_m \in \mathcal{U}$ of (non-random) graphs, the random variables

$$s(F_1, G_n), \ldots, s(F_m, G_n)$$ converges jointly in distribution

(iii) For every (non-random) $F \in \mathcal{U}$ the random variable $s(F, G_n)$ converge in distribution

(iv) For every (non-random) $F \in \mathcal{U}$ the $\mathbb{E}(s(F, G_n))$ converges.

- (ii), (iii), (iv) $\equiv$ limit objects $\equiv \{s(F_i, \Gamma)\}_{i=1}^{n}$, $(s(F, \Gamma))$ $\mathbb{E}(s(F, \Gamma))$ & $\Gamma \in \mathcal{U}_\infty$ a.s.

Ex:- Above Theorem hold for $t(F, \cdot)$ & $t_{ind}(F, \cdot)$

Corollary:- $\{G_n, n \geq 1\}$ be a random unlabelled graphs such that $v(G_n) \xrightarrow{p} \infty$ & let $G \in \mathcal{U}_\infty$. Then following are equivalent.

(i) $G_n \xrightarrow{p} G$ (ii) $s(F, G_n) \xrightarrow{p} s(F, G)$ $\forall$ (non-random) $F \in \mathcal{U}$

(iii) $\mathbb{E}\, s(F, G_n) \to s(F, G)$ $\forall$ non-random $F \in \mathcal{U}$.

Ex: Above Corollary hold for $t(F, \cdot)$ & $t_{ind}(F, \cdot)$

$E(t(F, G))$ — moments $E(t_{ind}(F, G))$ — cumulant.

Convergence to infinite graphs

- $\mathbb{N} = \{1, 2, 3, \dots\}$ – labeled infinite graphs $\equiv \mathcal{L}_\infty$
- – graphs in $\mathcal{L}_\infty$ are determine by their edge sets

$K_\infty \equiv$ complete graph on $\mathbb{N}$

$E(K_\infty) \equiv$ edge set

$\mathcal{L}_\infty \overset{\text{identification}}{\longleftrightarrow} \mathcal{P}(E(K_\infty))$

$\longleftrightarrow \{0,1\}^{E(K_\infty)}$

– Product – topology
– Compact metric space

- $n \geq 1$ $\quad \mathcal{L}_n$ $\quad [n]$ – labeled graph $\hookrightarrow \mathcal{L}_\infty$

$G \in \mathcal{L}_n \longleftrightarrow$ add $n+1, n+2, \dots$ isolated vertices $\hookrightarrow \mathcal{L}_\infty$.

$$\mathcal{L}_n \longleftarrow H|_{[n]} \longleftarrow H \in \mathcal{L}_\infty$$

induced graph on $[n]$

- G – finite graph – $\hat{G}$ be the random labeled graph obtained by a random labeling of the vertices G by numbers $1, \dots, v(G)$.

– G is labelled, ignore the labels, and randomly relabel
– $\hat{G}$ – random graph $v(\hat{G}) = v(G) \hookrightarrow$ random element in $\mathcal{L}_\infty$

$\hat{G} \equiv$ same notation := also for a random (finite) graph G given a random labelling

Theorem 4:- let $(G_n)_{n\geq 1}$ be a sequence of random graphs in $\mathcal{U}$ and assume that $v(G_n) \xrightarrow{p} \infty$. TFAE

(i) $G_n \xrightarrow{d} \Gamma$ in $\overline{\mathcal{U}}$ for some random $\Gamma \in \overline{\mathcal{U}}$

(ii) $\hat{G}_n \xrightarrow{d} H$ in $\mathcal{L}_\infty$ for some random $H \in \mathcal{L}_\infty$

If the above is true then $\mathbb{P}(H|_{[k]} = F) = \mathbb{E}(t_{ind}(F,\Gamma))$

$\forall\ F \in \mathcal{L}_k$. Furthermore $\Gamma \in \mathcal{U}_\infty$ a.s.

Proof:- G be a labelled graph

let $\hat{G}|_{[k]}$ & $k \leq v(G)$.

This random graph $\equiv$ equals $\equiv$ $G[k]' = G(v_1', \dots, v_k')$

Ex: $v_1', \dots v_k'$ are sampled at random without replacement from $v(G)$.

as $t_{ind}(F,G) = \mathbb{P}(F = G[k]')$

$\forall\ F \in \mathcal{L}_k \qquad \mathbb{P}(\hat{G}|_{[k]} = F) = t_{ind}(F,G)$, if $k \leq v(G)$

Apply the above to the random G_n

$$\mathbb{E}\, t_{ind}(F, G_n) \leq \mathbb{P}(\hat{G}_n|_{[k]} = F) \leq \mathbb{E}\, t_{ind}(F, G_n) + \mathbb{P}(v(G_n) \leq k)$$

$(*)$

- $v(G_n) \xrightarrow{p} \infty \implies \mathbb{P}(v(G_n) < k) \longrightarrow 0$ as $n \to \infty$ — $(*')$
- $G_n \xrightarrow{d} \Gamma$ in $\overline{\mathcal{U}}$, Theorem 3, $\iff \mathbb{E}\, t_{ind}(F, G_n) \longrightarrow \mathbb{E}(t_{ind}(F,\Gamma))$ as $n \to \infty$

$(**)$

$(*)$ and $(**)$ $(*')$ $\implies$

$$\implies \lim_{n\to\infty} \mathbb{P}(\hat{G}_n|_{[k]} = F) = \mathbb{E}(t_{ind}(F, \Gamma)) \quad - (\#)$$

$\forall F \in \mathcal{L}_k, k \geq 1$

$\mathcal{L}_k$ - finite set
$(\#)$

$\hat{G}_n|_{[k]} \xrightarrow{d} H_k$ for some random graph $H_k \in \mathcal{L}_k$

& $\mathbb{P}(H_k = F) = \mathbb{E}(t_{ind}(F, \Gamma)) \quad \forall F \in \mathcal{L}_k.$

$\mathcal{L}_\infty$ - product topology $\implies$ $\hat{G}_n \xrightarrow{d} H$ in $\mathcal{L}_\infty$ for some random H in $\mathcal{L}_\infty$ with $H|_{[k]} = H_k.$

(i) $\implies$ (ii) ✓

Conversely $\hat{G}_n \xrightarrow{d} H$ in $\mathcal{L}_\infty$

$$\implies \hat{G}_n|_{[k]} \xrightarrow{d} H|_{[k]}$$

as above $\mathbb{E}\, t_{ind}(F, G_n) = \mathbb{P}(\hat{G}_n|_{[k]} = F) + \alpha_k$

$\alpha_k \leq \mathbb{P}(v(G_n) < k)$

$$\xrightarrow[\forall k \geq 1]{n \to \infty} \mathbb{P}(H|_{[k]} = F)$$

$\Rightarrow$ by Theorem 3 $\exists\ \Gamma \in \mathcal{U}_\infty$ such that

$G_n \xrightarrow{d} \Gamma$ & $\mathbb{E}(t_{ind}(F, \Gamma)) = \mathbb{P}(H|_{[k]} = F)$

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5. Exchangeable Random Graphs

- A random infinite graph $H \in \mathcal{L}_\infty$ is exchangeable if its distribution is invariant under every permutation of its vertices

Ex:- above is equivalent to consider only finite permutations. i.e. $\begin{cases} \sigma(i) = i & \forall\ i > N_0 \text{ for some } N_0 \\ \sigma: \mathbb{N} \to \mathbb{N} & \text{1-1 onto} \end{cases}$

- $X_{ij} = \mathbb{1}((i,j) \in E(H))$, then $\{X_{ij}\}$, $1 \leq i, j \leq \infty$ is jointly exchangeable.

Lemma A :- H be a random graph in $\mathcal{L}_\infty$

T.F.A.E.

(i) H is exchangeable

(ii) $H|_{[k]}$ has a distribution invariant under all permutations of $[k]$, $\forall k \geq 1$

(iii) $\mathbb{P}(H|_{[k]} = F)$ — depends only on the isomorphism type of F and hence is a function of F as an unlabelled graph in $\mathcal{U}_k$ $\forall k \geq 1$.

Proof: (i) $\Rightarrow$ (ii) trivial

(ii) $\Leftrightarrow$ (iii)

(ii) $\Rightarrow$ (i) :- $\sigma : \mathbb{N} \to \mathbb{N}$ 1-1 onto & $\exists N_0 : \sigma(i) = i$ $\forall i > N_0$

Then $\sigma|_{[k]}$ is a permutation for $k > N_0$.

$H \cdot \sigma \equiv$ H with vertices permuted by σ

$k > N_0$: $\quad H \cdot \sigma|_{[k]} = H|_{[k]} \cdot \sigma \overset{d}{=} H|_{[k]}$ by (ii)

$\Rightarrow H \cdot \sigma \overset{d}{=} H$ $\square$