# Recall:

- $H \in \mathcal{L}_\infty$ is exchangeable if its distribution is invariant under every permutation of its vertices.

**Result**: • $H$ is exchangeable

$\iff$ $H|_{[k]}$ has a distribution invariant under permutations of $[k]$, $\forall k \geq 1$

$\iff$ $\mathbb{P}(H|_{[k]} = F)$ depends only on the isomorphism type of $F$.

• The limit $H$ in Theorem 4 was exchangeable

$$\mathbb{P}(H|_{[k]} = F) = \mathbb{E}\, t_{ind}(F, \Gamma) \leftarrow \left[ G_n \xrightarrow{d} \Gamma \text{ in } \overline{\mathcal{U}} \iff \hat{G}_n \xrightarrow{d} H \text{ in } \mathcal{L}_\infty \right]$$

**Thm 5**. There is a 1-1 correspondence between distributions of random elements of $\Gamma \in \mathcal{U}_\infty$ and distributions of exchangeable random graphs $H \in \mathcal{L}_\infty$.

$$\mathbb{E}\, t_{ind}(F, \Gamma) = \mathbb{P}(H|_{[k]} = F)$$

$F$ was on $[k]$ vertices. $- \in \mathcal{L}_k$

$$\mathbb{E}\, s(F, \Gamma) = \mathbb{P}(H \supset F) \quad \forall F \in \mathcal{L}$$

$$H|_{[n]} \xrightarrow{d} \Gamma \quad \text{as } n \to \infty$$

Corollary 5 :- There is a 1-1 correspondence between $U_\infty$

and

the extreme points of the set of distributions of exchangeable random infinite graphs $H \in \mathcal{L}_\infty$.

The correspondence is given by

$$\Gamma \longrightarrow \cancel{t}(F, \Gamma) = \mathbb{P}(H \supset F) \longrightarrow H$$

$\forall F \in \mathcal{L}$

Further, $H|_{[n]} \xrightarrow{a.s} \Gamma$ in $\bar{U}$ as $n \to \infty$

- [Proof in Diaconis - Janson - has a typographical error]

**Proof** :- $\mathcal{M} \equiv$ set of distributions of exchangeable random infinite graphs $H \in \mathcal{L}_\infty$.

- **Result** :- Extreme points of $\mathcal{M}$ are delta measures in $\mathcal{M}$

$\Rightarrow$ Extreme points are in 1-1 correspondence with elements of $\mathcal{L}_\infty$.

Theorem 5 $\Rightarrow$ $H \in \mathcal{L}_\infty \overset{1-1}{\longleftrightarrow} \Gamma \in \mathcal{U}_\infty$

extreme points of $\mathcal{M}$ are in 1-1 correspondence with elements of $\mathcal{U}_\infty$.

Using Theorem 5 – Distribution as delta-measures so correspondence becomes

$$t(F, \Gamma) = \mathbb{P}(H \supset F) \quad \forall F \in \mathcal{L}.$$

Similarly from Theorem 5:

$$H|_{[n]} \xrightarrow{a.s.} \Gamma \text{ in } \bar{\mathcal{U}} \text{ as } n \to \infty.$$

$\square$

Theorem 6: let $H$ be an exchangeable random infinite graph. Then the following are equivalent

(i) The distribution of $H$ is an extreme point in the set of exchangeable distributions in $\mathcal{L}_\infty$

(ii) If $F_1$ and $F_2$ are two (finite) graphs with disjoint vertex sets $V(F_1)$, $V(F_2) \subseteq \mathbb{N}$

then
$$\mathbb{P}(H \supseteq F_1 \cup F_2) = \mathbb{P}(H \supseteq F_1)\,\mathbb{P}(H \supseteq F_2)$$

(iii) The restrictions of $H|_{[k]}$ and $H|_{[k+1,\infty)}$ are independent $\forall k$.

(iv) let $\mathcal{F}_n$ be a $\sigma$-field generated by $H|_{[n,\infty)}$

Then the tail $\sigma$-field

$\bigcap_{n=1}^{\infty} \mathcal{F}_n$ is trivial, i.e.

contains only event with probability 0 or 1

**Proof:** will do later.

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- Connect [LS 06]

$G_n \longrightarrow K$

in $d_{sub}(,)$ metric.

Diaconis

$\longleftarrow - - - - \longrightarrow$

Janson

Aldous / Theory

Extension of Definiti Theorem to arrays

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$W_s =$ graphons $:= \{ W : [0,1]^2 \to [0,1]$ symmetric and measurable $\}$

Done earlier [LLN construction]

$W_s \ni W$

$[n] = \{1, .. n\}$

$\uparrow$ $u_1$ ... $\uparrow$ $u_n$ $\sim$ Uniform(0,1)

$\underline{G(n,W)}$ : - $i \sim j$ w.p. $W(U_i, U_j)$

$i, j \in [n]$

__Extension:__

$G(\infty, W) \equiv$ Vertex set $\mathbb{N}$

$i \sim j$ wp. $W(U_i, U_j)$

$i, j \in \mathbb{N}$

$\Rightarrow G(\infty, W) \big|_{[n]} = G(n, W)$

Ex:- $G(\infty, W)$ is an exchangeable infinite random graph.

$$X = \{X_{ij} = \mathbb{1}(i \sim j \in \text{Edge in } G(\infty,W)\}_{i,j}$$

– jointly exchangeable

[Aldous-Hoover] $\exists\, \mu$ $\quad \mathbb{P}(X \in A) = \int \mathbb{P}_W(A)\, \mu(dW)$

Take $W \in \mathcal{W}_0$ – $G(\infty, W)$ is an exchangeable random graph

Theorem 6 (ii) holds for $G(\infty, W)$

$V(F_1) \cap V(F_2) = \phi$ & $V(F_i) \subseteq \mathbb{N}$ $\quad \mathbb{P}(G(\infty,W) \supseteq F_1 \cup F_2) = \mathbb{P}(G(\infty,W) \supseteq F_1)\, \mathbb{P}(G(\infty,W) \supseteq F_2)$

$\Downarrow$

Theorem 6 (i) holds $\Leftarrow$ Distribution of $G(\infty,W)$ is an extreme point

$\Downarrow$

Corollary 5 – $G(\infty, W) \longrightarrow \Gamma_W \in \mathcal{U}_\infty$ – (1-1 correspondence)

$\Downarrow$

Theorem 5 – $G(\infty, W)\big|_{[n]} \xrightarrow{a.s} \Gamma_W \quad n \to \infty$

$G(n, W) \xrightarrow{a.s} \Gamma_W$ as $n \to \infty$

F on [k]

$$s(F, \Gamma_W) = \mathbb{P}(F \subseteq G(k, W)) = E\Big[\prod_{(i,j)\in E(F)} W(U_i, U_j)\Big]$$

$$(*)\quad = \int_{[0,1]^k} \prod_{(i,j)\in E(F)} W(x_i, x_j) \prod_{i=1}^{k} dx_i$$

[LS 06] $\mathcal{U}_\infty \equiv$ limit points of finite graphs are in fact $\Gamma_W$ that satisfy $(*)$

## Proof of Theorem 6

$H$ - exchangeable

(i) The distribution of $H$ is an extreme point in the set of exchangeable distributions in $\mathcal{L}_\infty$

↓ Corollary 5

$\exists\ \Gamma \in \mathcal{U}_\infty$

$$(*) — \mathbb{P}(H \supseteq F) = \delta(F, \Gamma) \quad \forall F \in \mathcal{L}.$$

$F$ - labelled graph $1, \dots, k$ $\quad k = v(F)$

but $(*)$ is invariant under labelling.

$\Rightarrow$ $(*)$ hold for all $F$ such $V(F) \subseteq \mathbb{N}$

finite graph

$\Rightarrow$ $\Gamma$ is an element of $\mathcal{U}_\infty$

$$\delta(F_1 \cup F_2, \Gamma) = \delta(F_1, \Gamma)\, \delta(F_2, \Gamma)$$

[Direct sum lemma]

$\Gamma$-random

$$E\Big(\prod_{i=1}^{n} \delta(F_i, \Gamma)\Big) = E\Big[\delta\Big(\bigoplus_{i=1}^{n} F_i, \Gamma\Big)\Big]$$

$$\mathbb{P}(H \supseteq F_1 \cup F_2) \overset{(*)}{=} \delta(F_1 \cup F_2, \Gamma)$$

$$= \delta(F_1, \Gamma)\, \delta(F_2, \Gamma)$$

$$\overset{(*)}{=} \mathbb{P}(H \supseteq F_1)\, \mathbb{P}(H \supseteq F_2)$$

(ii) $F_1$ and $F_2$ are two finite graphs

$V(F_1) \cap V(F_2) = \phi \qquad V(F_i) \subseteq \mathbb{N}$

$$\mathbb{P}(H \supseteq F_1 \cup F_2) = \mathbb{P}(H \supseteq F_1)\, \mathbb{P}(H \supseteq F_2) \quad — (\dagger)$$

Earlier $t_{ind}(F, G) = \mathbb{P}(F = G[k]') \quad \ell \geq 1$

$\Rightarrow \quad V(F_1) = \{1, \dots k\} \qquad V(F_2) \subseteq \{k+1, k+2, \dots, \ell\}$

$$H|_{[k]} = F_1 \quad \text{and} \quad H|_{[k+1, \ell]} = F_2$$

(by (+)) are independent $\forall\ \ell \geq 1$

(iii) The restriction of $H|_{[k]}$ and $H|_{[k+1, \infty)}$ are independent $\forall\ k \geq 1$

$\square$