Sparse Random Graphs : Assignment 4

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Submit solutions to the below problems via Moodle by 26th December 10:00 PM.

1. Define the n-hypercube graph as follows : $V_n = \{0,1\}^n$ is the vertex set and edge set is $E_n = \{(v,w) : v - w = \overset{+}{-} e_k \text{for some } 1 \leq k \leq n\}$ i.e., (v,w) is an edge if they differ exactly at one co-ordinate. Let $H(n,p)$ denote the random graph such that each edge in E_n is chosen with probability $p \in [0,1]$ independently of each other. Let $O := (0,\ldots,0) \in V_n$ for all $n \geq 1$. For any $v \in V_k$, we set $v(n) := (v,0,\ldots,0) \in V_n$ for all $n \geq k$. Set $p_n = \min\{c/n, 1\}$ for all $n \geq 1$ where $c \in (0,\infty)$ and $H_n := H(n,p_n)$. What is the local weak limit of H_n ? First, describe the main steps in the proof and then give details for the individual steps.

2. Let (G,w,o) be a unimodular random weighted rooted graph with $w \in \mathbb{W}$, a Polish space. Let $A \subset \mathcal{G}_*(\mathbb{W})$ be an event invariant under re-rooting i.e., if $(G,w) \cong (G',w)$, then $(G,w,o) \in A$ iff $(G',w,o) \in A$. Assume that $\mathbb{P}(A) > 0$. Define a randomly weighted rooted graph (H,o) by the following probability distribution given by
$$\mathbb{P}((H,w',o) \in \cdot) = \mathbb{P}((G,w,o) \in .|(G,w,o) \in A).$$
Show that (H,w',o) is unimodular.

3. We define an end of a rooted tree (T,o) as a self-avoiding, semi-infinite path on T starting at o . Let (G,o) be a.s. an infinite connected and unimodular random graph. Show that if $\mathbb{E}[deg_G(o)] = 2$ then show that (G,o) is a.s. a tree and it has one or two ends a.s..

ADDITIONAL PROBLEMS (for practice) :

1. Show that sofic graphs are unimodular via Lusin's theorem.

2. Let $\mathbb{W}$, a Polish space be the mark space. Show that $\mathcal{G}_*(\mathbb{W}), \mathcal{G}_{**}(\mathbb{W})$ are complete seperable metric spaces.

3. Show that $(G,u,v) \mapsto (G,u)$ is a continuous map from $\mathcal{G}_{**}(\mathbb{W})$ to $\mathcal{G}_*(\mathbb{W})$.

4. Let (G,w,o) be a unimodular random weighted rooted graph with $w \in \mathbb{W}$, a Polish space. Suppose that $\phi : \mathcal{G}_{**}(\mathbb{W}) \to \mathbb{R}_+$ be a measurable function on the space of doubly rooted graphs. Set $w'(u,v) = \phi(G,u,v)$ for $(u,v) \in E(G)$. Show that (G,w',o) is a unimodular random weighted rooted graph.

5. Let (G, w, o) be a unimodular random weighted rooted graph with $w \in \mathbb{W}$, a Polish space. Let $B \subset \mathbb{W}$ be a Borel subset. Define a new weighted graph (G', w') where $V' = V$ and edge-set $E' = \{(u, v) : w(u, v) \in B, w(v, u) \in B\}$ and $w'(u, v) = w(u, v)$ for $(u, v) \in E'$. Show that (G', w', o) is a unimodular random weighted rooted graph.

6. Show that if G is an infinite graph, each connected component of $MSF(G)$ is infinite.