# 8/12 – L14 – MST COMPUTATIONS ON GW TREES

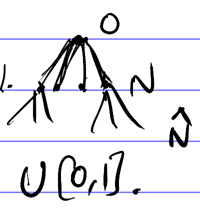
$GW(P)$ – $P = (P(k))_{k\geq 0}$ – pmf of the offspring random variable $N$.

Assume $P(0) = P(1) = 0$.

$$EN^2 = \sum_{k\geq 2} k^2 P(k) < \infty.$$

$(T,0) \stackrel{d}{=} UGW(P)$

$P(\hat{N} = k-1) = \frac{kP(k)}{E[N]}$, $k \geq 1$.




$w = (w(e))_{e\in (T,0)}$ i.i.d. $U[0,1]$.

$$\zeta_*(T) = \frac{1}{2} E \sum_{v\in V} w(0,v) \mathbb{1}[(0,v) \in MSF(T)]$$

$$MSF(T) = MSF((T,w)).$$

[Assume $\exists G_n \ni (G_n, w) \xrightarrow{LWC} (T,0,w)$ & $\frac{\zeta(G_n)}{|V_n|} \longrightarrow \zeta_*(T)$. ]

$$2\zeta_* = \sum_{k\geq 1} kP(k)\, E\big[w(0,1)\mathbb{1}[(0,1)\in MSF(T)] \,\big|\, N=k\big]$$

$$= \sum_{k\geq 1} kP(k) \int_0^1 t\, \mathbb{P}\big[\{0,1\} \in MSF(T) \,|\, w(0,1)=t, N=k\big]dt \quad \text{— ①}$$

$T'_t$ $= T - \{0,1\}$, $\hat{T}_t$

$$\big\{\{0,1\} \notin MSF(T)\big\} = \{|T'_t| = \infty, |\hat{T}_t| = \infty\}$$

$T'_t = \{e \in T' : w(e) \leq t\}$ — $\hat{T}$ & $T'$ are independent.

$\hat{T}_t = \{e \in \hat{T} : w(e) \leq t\}$

$$① = \sum_{k\geq 1} kP(k) \int_0^1 t\big(1 - \mathbb{P}(|\hat{T}_t| = \infty)\, \mathbb{P}(|T'_t| = \infty \,|\, N = k-1)\big)dt$$

$$= \sum_{k\geq 1} kP(k) \int_0^1 t\big\{1 - \hat{q}(t)\, \mathbb{P}(|T'_t| = \infty \,|\, N=k-1)\big\}dt \quad \text{— ②}$$

$\hat{q}(t) = 1 - \hat{p}(t)$, $\hat{p}(t) = p_{ext}(\hat{T}_t)$

MTP $\Rightarrow$

$$\sum_{k\geq 1} kP(k)\mathbb{P}(|T'_t| = \infty \,|\, N = k-1) = EN \cdot \hat{q}(t)$$

$\varphi'(1) = EN = \sum_{k\geq 1} kP(k)$; $\varphi(x) = E[x^N]$

$\hat{\varphi}(x) = \varphi'(x)/\varphi'(1)$ — $\hat{T}$

① + ② $\Rightarrow$

$$2\zeta_* = \varphi'(1) \int_0^1 t\,(1 - \hat{q}(t)^2)\,dt$$

$\hat{p}(t)$, smallest root $r \ni \hat{\varphi}_t(r) = r$

$\hat{q}(t) = 1 - \hat{\varphi}_t(1 - \hat{q}(t))$ $\quad \hat{\varphi}_t = E[x^{\hat{N}_t}]$

$= 1 - \hat{\varphi}(1 - t q(t))$

. . . . .

$$\Rightarrow \zeta_* = -\varphi'(1) \int_0^1 (1-x) \ln(1 - \hat{\varphi}(x))\,dx$$

[See lemma 3.7 of Bordenave "Counting unimodular..." for full details ]

1. $(T,0) = T_d$ – d-regular tree. $\varphi'(1) = d$, $\hat{\varphi}(x) = x^{d-1}$

$\Rightarrow \zeta_*(T_d) = -d\int_0^1 (1-x)\ln(1-x^{d-1})\,dx.$

2. $(T,0) \stackrel{d}{=} BGW(Poi(\lambda))$, $\hat{\varphi} \equiv \varphi$, $\varphi'(1) = \lambda$

$$\zeta_*(\lambda) = -\lambda \int_0^1 (1-x)\ln(1 - e^{-\lambda(1-x)})\,dx$$

$$= -\int_0^1 (\lambda x)\ln(1 - e^{-\lambda x})\,dx$$

$$\zeta_*(1) = \int_0^1 t(1 - \hat{q}(t)^2)\,dt$$

$\hat{T} \stackrel{d}{=} T \stackrel{d}{=} BGW(Poi(1))$ $\quad T_t \stackrel{d}{=} \hat{T}_t \stackrel{d}{=} BGW(Poi(t))$ ↳ by thinning property of Poisson r.v.

$\hat{q}(t) = q(t) = 1 - p_{ext}(BGW(t))$

$= 1 - e^{-t(1-q(t))}$

$$\frac{1}{2}\int_0^\infty t(1 - q(t)^2)\,dt = \zeta(3) \longrightarrow \text{Riemann-Zeta fn.}$$

**Eg:** $K_n$ – complete graph. $w(e)$ – i.i.d. $U[0,1]$

$E[\zeta(K_n)]$.

$K_n(t) = (K_n, \{e : w(e) < t\}) = G(n,t) \stackrel{d}{=} ER(n,t)$, $t \in (0,1)$.

$(u,v) = e \in MST(K_n) \Leftrightarrow u \not\leftrightarrow v$ in $G(n, w(e))$

$$E[\zeta(K_n)] = \sum_{e\in K_n} \int_0^1 t\, P(u \not\leftrightarrow v \text{ in } G(n,t))\,dt$$

$(t = s/n)$

$$\approx \frac{1}{n^2} \sum_{e\in K_n} \int_0^{c\log n} s\, \mathbb{P}(u \not\leftrightarrow v \text{ in } G(n, s/n))\,ds$$

[$\mathbb{P}(G(n,t) \text{ is connected}) \to 1$ if $t = \frac{\log n + b_n}{n}$, $b_n \to \infty$]

$$\approx \frac{1}{n^2}\sum_{e\in K_n} \int_0^\infty s\, \mathbb{P}(u \not\leftrightarrow v \text{ in } G(n, s/n))\,ds$$

$$= \frac{1}{2}\int_0^\infty \mathbb{P}(u \not\leftrightarrow v \text{ in } G(n, s/n))\,ds$$

$G(n, s/n)$ $\stackrel{\shortparallel}{}$

$(G_n(s), u) \longrightarrow (T_s, u)$ $\quad T_s = GW(Poi(s))$

$(G_n(s), v) \longrightarrow (T'_s, v)$ $\quad T'_s \stackrel{d}{=} T_s$

$P(v \in B_r^{G_n(s)}(u)) \to 0 \quad \forall r \geq 1$

$\Rightarrow T_s$ & $T'_s$ are indep. & disjoint $GW(Poi(s))$ trees

$\Rightarrow \{(u,v) \notin MSF(G(n, s/n))\}$

$\approx \{(u,v) \notin MSF(G_n(s))\}$

$= \{|T_s| = |T'_s| = \infty\}$

$\Rightarrow \mathbb{P}((u,v) \notin MSF(G(n, s/n))) \to \mathbb{P}(|T_s| = \infty)^2$

$= q(s)^2$

Frieze 1985. $E[\zeta(K_n)] \longrightarrow \frac{1}{2}\int_0^\infty s(1 - q(s)^2)\,ds = \zeta(3)$.

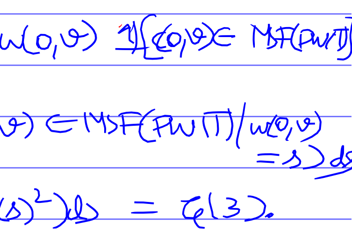
LWC Proof: $(K_n, w)$ $w(e)$ i.i.d. $EXP(1)$.

$w_n(e) = n w(e) \stackrel{d}{=} EXP(n)$.

$K'_n = (K_n, w_n)$ $\quad \frac{1}{n}\zeta(K'_n) = \zeta(K_n)$.

$K'_n \xrightarrow{LWC} PWIT(1)$

Poisson weighted ∞ tree.




$S$ is a Poisson process on $\mathbb{R}_+$ with mean 1

i.e., $S_{i+1} - S_i$ are i.i.d. $EXP(1)$ r.v.'s.

$$E[\zeta_*(PWIT)] = \frac{1}{2} E\Big[\sum_{v\in V} w(0,v)\mathbb{1}[(0,v) \in MSF(PWIT)]\Big]$$

[See Pg 125 & 126 for more details]

$$= \frac{1}{2}\int_0^\infty s\, \mathbb{P}((0,v) \in MSF(PWIT) \,|\, w(0,v) = s)\,ds$$

$$= \frac{1}{2}\int_0^\infty s\,(1 - q(s)^2)\,ds = \zeta(3).$$

**Big message:** Toy model for combinatorial optimization problems. Put i.i.d. $EXP(n)$ weights on $K_n$.

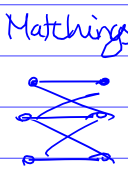
$\frac{\alpha_n}{n} = \frac{1}{n}\min_{H \subseteq K_n} \alpha(H) \longrightarrow \alpha(PWIT)$

$\to$ $G(n,p)$ – $|M|$ – size of maximal matching

$\frac{|M(G(n,p))|}{n} \longrightarrow$ ? ✓ LWC theory in Ch. 4 of Bordenave's notes.

Qns: $\frac{|IM(G(n,p))|}{n} \longrightarrow$ ??? $\quad$ IM – Induced matching

Matchings on Hypergraphs.



i.i.d. weights $U(0,1)$ $\quad E[\text{min cost matching}] \to \zeta(2)$

. . . .

. . . . $\quad$ . . . . $\quad E[\quad] \to$ ???

# SUMMARY.

- ER graph has many regimes
- It "looks like" a tree "locally" in the sparse regime
- Formalized via LWC - Extends to weighted graphs.

[other Eg: Configuration Model, Hypercube percolation]
(See any of references) Assignment

- Limits are Unimodular RG.
- LWC ⟹ bounded fns on bounded degree graphs are estimable!

⟹ Size of largest conn. component under some assumptions

⟹ Aldous-Steele Cty thm for MST.
When limiting tree is GW, we can compute the limit.

See Bordenave - "Counting - - - " for more eg. of non-trivial estimable functions.

Is Spec ($A_G$) estimable ?
Adj. matrix

## LWC Link:

$$(K_n, w) \xrightarrow{LWC} ?$$

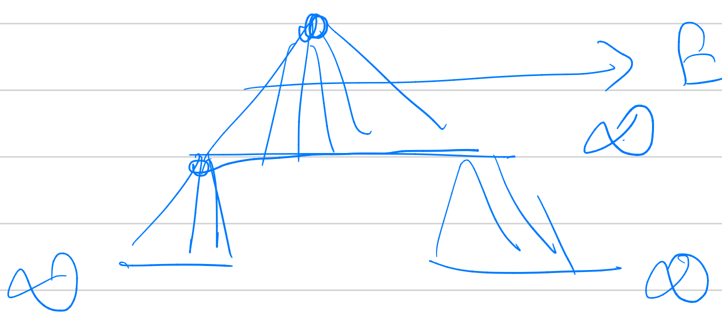
let $w(e)$ i.i.d. $EXP(1)$.

$$\mathbb{P}(w(e) \geq x) = e^{-x}, \quad x \geq 0$$

$$\mathbb{P}(w_n(e) \geq n^{-1}x) = e^{-x}, \quad w_n(e) = n\,w(e).$$

$\Rightarrow$ $w_n(e)$ i.i.d. $EXP(n)$

$$(K_n, w_n) \xrightarrow{LWC} PWIT(x)$$



Edge weights $\stackrel{d}{=} \underbrace{\{S_1, S_2, \dots\}}_{= \, S}$

$S$ = Poisson process on $\mathbb{R}_+$

i.e., $S_0 = 0$, $S_{i+1} - S_i$ are i.i.d. $EXP(1)$.

$$N_t = |S \cap [0,t]| \stackrel{d}{=} Poi(t) \text{ r.v.}$$

$$\overset{\stackrel{d}{=}}{\{U_1, \dots, U_{N_t}\}}, \quad U_i \text{ i.i.d. } U[0,t]$$

$$\mathbb{E}\left[T_x(PWIT)\right] = \frac{1}{2}\mathbb{E}\sum_{v \in V} w(0,v)\, \mathbb{1}\left[(0,v) \in MSF(PWIT)\right]$$

$$= \frac{1}{2}\int_0^\infty s\, \mathbb{P}\big((0,v) \in MSF(PWIT) \,\big|\, w(0,v) = s\big)\, ds$$

$$w(0,v) = s$$

$$(0,v) \in MST(PWIT) \Leftarrow |(PWIT(s),0)| < \infty \text{ or } |(PWIT(s),v)| < \infty$$

$$\Rightarrow \mathbb{P}((0,v) \in MSF(PWIT)) = (1 - \mathbb{P}(|PWIT(s),0)| = \infty)^2)$$

$$(PWIT(s),0) \overset{d}{=} GW(Poi(s))$$

since $|S_n(0,s]| \overset{d}{=} Poi(s)$.

$$\tau((K,w)) = \frac{1}{n}\tau((K_n,w_n))$$

$$\mathbb{E}[\tau((K,w))] = \frac{1}{n}\mathbb{E}[\tau((K_n,w_n))] \to \mathbb{E}[\tau_*(PWIT)]$$

$$= \zeta(3).$$