Random graph &
Limits

M2 :- SEP -DEC 2020

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Course Progress : Sparse Random Graphs

This page is to maintain topics covered in the sparse random graphs class.

L1 - Sep 1 : Metric space of rooted graphs. Simple examples and continuous functionals.

L2- Sep 8 : Erdos-Renyi random graphs. Sparse regime. Subgraph counts.

L3- Sep 10 : Poisson approximation of cycle counts. Bienayme-Galton-Watson tree basics.

L4- Sep 22 : Breadth-first exploration for Bienayme-Galton-Watson tree and Erdos-Renyi graph.

L5- L6- Sep 29 : Local structure of the root in Erdos-Renyi graph. Basics of weak convergence on metric spaces. Definition of local weak convergence.

L6- Oct 6 : Criterion for local weak convergence ; Local weak convergence of random graphs. Completeness of limits. Local weak convergence of Erdos-Renyi random graph.

L7- Oct 13 : Locality of giant component ; Properties of the giant ; Discussion of Assignments 1 and 2.

L8- Oct 20 : Vertex structure of the giant component ; Giant component of the ER graph ; Small-worldness of the ER random graph.

L9- Oct 27 : Tightness in metric spaces. Compact sets for local weak topology. Criteria for tightness of random graphs. Estimable functions.

Nov 3 : BREAK

L10 - Nov 10 : Unimodular random graphs. Examples.

L11 - Nov 17 : Soficity and Unimodularity. Involution invariance. Unimodular Galton-Watson trees.

L12 - Nov 24 : Doubly rooted graphs and Local weak convergence of weighted graphs ; Unimodularity of weighted graphs ; Discussion on other random graph examples.

L13 - Dec 1 : Minimal spanning trees - Definition and basic properties. Aldous-Steele continuity theorem.

L14- Dec 8 : Explicit computations for Galton-Watson trees. Idea of Frieze's $\zeta(3)$ theorem. Summary of the course.

01/09/20: LECTURE 1 : Metric space of rooted graphs

$G = (V, E)$ - graphs. V - vertex set, countable.

E - Edge set $\subseteq V \times V$
(neighbourhood of v) unordered pairs.

$v \sim w$ if $(v, w) \in E$. $N_G(v) = \{ w \in V : v \sim w \}$
neighbour, adjacent } $\deg_G(v) = \deg(v) = |N(v)|$

<u>locally finite</u> if $\deg_G(v) < \infty$, $\forall v \in V$.

<u>Rooted graph</u> : (G, o) is a rooted graph if G is a graph & $o \in V(G)$. o is called the root.

d_G - graph distance; $d_G(u, v)$ = length of shortest path between u & v (= (no. of edges))
$\rightarrow (G, d_G)$ is a metric space.

$H \subseteq G$ is a subgraph if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$.

$H = (V_1, E_1) \subseteq G$ is an induced subgraph if

$$E_1 = (V_1 \times V_1) \cap E(G)$$

i.e., all edges in G between vertices of H are present in H. Induced subgraphs are specified by vertex set alone.

$$B_r(v) = B_r^{(G)}(v) = \{w : d_G(v, w) \le r\}$$

We also use $B_r(v)$ to denote the induced subgraph on $B_r(v)$. i.e.,

$$E(B_r(v)) = \{(u, w) \in E : d_G(v, u) \le r, \; d_G(v, w) \le r\}$$

<u>Defn</u>: Graph isomorphism

$G_1 \cong G_2$ (G_1 is isomorphic to G_2) if $\exists$ a bijection $\phi : V_1 \to V_2 \ni$ $(i, j) \in E_1 \Leftrightarrow (\phi(i), \phi(j)) \in E_2$. ϕ is called graph isomorphism. ($\phi : G_1 \to G_2$, notation for simplicity)

$(G_1, o_1) \cong (G_2, o_2)$ if $\exists$ a graph isomorphism $\phi : G_1 \to G_2 \ni \phi(o_1) = o_2$.

$\phi : G_1 \to G_2$ is a graph homomorphism if $\phi : V_1 \to V_2$ & $(i, j) \in E_1 \Rightarrow (\phi(i), \phi(j)) \in E_2$.

$\mathcal{G}_*$ = space of rooted connected graphs modulo isomorphisms

= set of Equivalence classes of rooted connected graphs

$[(G,o)] \in \mathcal{G}_*$ but we'll use notation $(G,o) \in \mathcal{G}_*$ keeping in mind that $(G',o') \cong (G,o)$ are same.

<u>DEFN:</u> Let $(G_1,o_1), (G_2,o_2) \in \mathcal{G}_*$.

$$R^* = \sup\{r \geq 0 : B_r^{(G_1)}(o_1) \cong B_r^{(G_2)}(o_2)\}$$

$(R^* \geq 0)$

$$d_{\mathcal{G}_*}((G_1,o_1),(G_2,o_2)) = \frac{1}{R^*+1}$$

PROPERTIES OF $(\mathcal{G}_*, d_{\mathcal{G}_*})$: $d_{\mathcal{G}_*}$ is well-defined!

1. $d_{\mathcal{G}_*}$ is a metric space

(Lemma A.9 of vdH-2)

- $d_{\mathcal{G}_*} \geq 0$
- $d_{\mathcal{G}_*} = 0 \iff R_* = \infty \iff (G_1,o_1) \cong (G_2,o_2)$

Propn A.8 of vdH-2 : [ULTRAMETRICITY].

$$d_{\mathcal{G}_*}\big((G_1,o_1),(G_3,o_3)\big) \le \max\Big\{ d_{\mathcal{G}_*}\big((G_1,o_1),(G_2,o_2)\big), d_{\mathcal{G}_*}\big((G_3,o_3),(G_1,o_1)\big) \Big\}$$

$\Rightarrow$ $(\mathcal{G}_*, d_{\mathcal{G}_*})$ is an ultrametric space

Sketch of proof: $R^*_{ij} = \sup\{r : B_r^{(G_i)}(o_i) \cong B_r^{(G_j)}(o_j)\}$

$R^*_{12} \geq \min\{R^*_{13}, R^*_{23}\} \Rightarrow \ldots\ldots$ ∎

LEMMA: $(\mathcal{G}_*, d_{\mathcal{G}_*})$ is seperable.

Proof: $d\big(B_r^{G}(o), (G,o)\big) \le \frac{1}{r+1} \to 0$ as $r \to \infty$.

Ex. $\Rightarrow$ seperability.

$S_* = \{$ set of all ~~finite~~ finite graphs in $\mathcal{G}_*\}$ (finite V)

S_* is countable as $\exists$ finitely many eq. classes on graphs with n vertices, $\forall n$.

Given (G,o), $B_r^{(G)}(o) \in S_*$
& $d\left((G,o), B_r^{(G)}(o)\right) \le \frac{1}{r+1}$. ∎

LEMMA A.11 of vdH-2.
Let $\{(G_r, o_r)\}_{r\ge 0}$ be connected loc. finite rooted graphs that are compatible i.e., $(B_r^{(G_s)}, o_s) \cong (B_r^{(G_r)}, o_r)$ $\forall\, r \le s$.

Then $\exists\,!$ (upto isomorphisms) (G,o)
$\ni$ $(G_r, O_r) \simeq (B_r^{(G)}, o)$

Propn A.10 of vdH-2 : $(\mathcal{G}_*, d_{\mathcal{G}_*})$ is a Polish space
A.6

Examples of convergent sequences:

(1) $(G,o) \in \mathcal{G}_*$ $\quad B_r^{G}(o) \longrightarrow (G,o)$

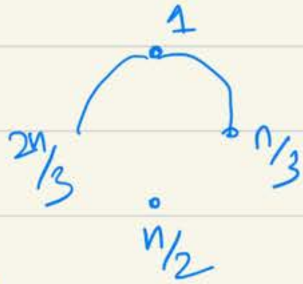
(2) $(G_n, 1) = (C_n, 1)$ – cycle graph

$(G_n, 1) \longrightarrow (\mathbb{Z}, 0)$.

$(\mathbb{Z},-1) \cong (\mathbb{Z},1)$

$$R^* = \sup \{ r : B_r^{(G_n)}(1) \cong B_r^{(\mathbb{Z})}(0) \}$$

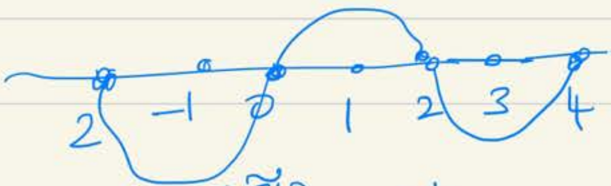
$$\geq \frac{n}{3}$$



$$\Rightarrow \quad d((G_n, 1), (\mathbb{Z}, 0)) \to 0.$$

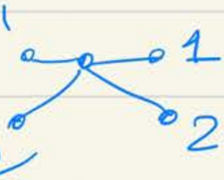
$(C_n, 1)$ looks "locally like" $(\mathbb{Z}, 0)$.

3) $(\mathbb{Z}, 0)$ $\quad (\tilde{\mathbb{Z}}, 0)$



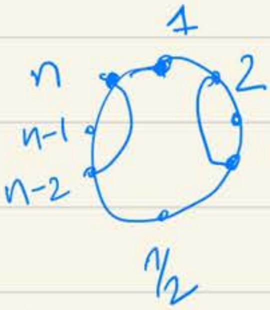
$$d((\mathbb{Z},0), (\tilde{\mathbb{Z}}, 0)) = 1$$

$B_1^{(\tilde{\mathbb{Z}})}(0) =$



4)

$(C_n, 1)$



$(\tilde{C}_n, 1)$. $\quad$ n large.

$n-2 > \frac{3n}{4}$, $\quad 4 < \frac{n}{4}$.

$$d((C_n, 1), (\tilde{C}_n, 1)) = \frac{1}{2}.$$

$$(\tilde{C}_n, \tfrac{n}{2}) \to R^*((\tilde{C}_n, \tfrac{n}{2}), (C_n, 1)) \geq \frac{n}{4}.$$

$$R^*((\tilde{C}_n, \tfrac{n}{2}), (\mathbb{Z}, 0)) \geq \frac{n}{4}.$$

$$(\tilde{C}_n, \tfrac{n}{2}) \longrightarrow (\mathbb{Z}, 0).$$

$$(\tilde{C}_n, v) \longrightarrow (\mathbb{Z}, 0) \quad \text{for "most" } v.$$

What we want to capture is for large n:

$$(\tilde{C}_n, v) \approx (C_n, 1) \quad \text{for "most" } v.$$

Ex. (1) (A) Fix $H_* \in \mathcal{G}_*$. $h : \mathcal{G}_* \longrightarrow \{0,1\}$

$$h(G,0) = \mathbb{1}[B_r(0) \cong H_*].$$

(2) $h : \mathcal{G}_* \longrightarrow \mathbb{N}$

$$(G,0) \longmapsto |B_r^G(0)|$$

$r=1 \Rightarrow |B_r^G(0)| = \deg(0) + 1.$

Are (1) & (2) cts ?

Ex* what about $(G,0) \longmapsto f(B_r^G(0))$?

$f :=$ some fn on finite, rooted connected graphs.

takes values in some Polish space.

For eg: $(G,0) \mapsto (B_r^G(0), 0) \in \mathcal{G}_*$ i.e, $f = \mathrm{Id}$.

or $f : \mathcal{G}_* \to \mathbb{R}$ bdd fn. ?

08|09|20. L2: Erdös-Rényi random graphs.

ER: $G(n,p)$, $p \in [0,1]$ if $p>1$, take $p \wedge 1$.

$V = [n] = \{1, \dots, n\}$

$E = \{(i,j) : X_{ij} = 1\}$ - unordered pairs.

where X_{ij}, $1 \le i < j \le n$ are i.i.d. Ber(p) rand. variables.

Also we take $X_{ij} = X_{ji}$, $1 \le i,j \le n$. $X_{ii} = 0$.

Undirected & simple graph.

The graph is formed by keeping edges in a complete graph K_n with prob p & each edge is kept independently of other edges.

Let G be a fixed graph on n vertices with e edges then

$$\mathbb{P}(G(n,p) = G) = p^e (1-p)^{\binom{n}{2} - e}.$$

If $p = \frac{1}{2}$, $\mathbb{P}(G(n,p) = G) = \left(\frac{1}{2}\right)^{\binom{n}{2}}$

i.e., $G(n, \frac{1}{2})$ is uniformly selecting a graph on n vertices. Let $\mathcal{G}_n$ - graphs on n vertices.

$\Rightarrow G(n, \frac{1}{2}) \stackrel{d}{=} \mathrm{Unif}(\mathcal{G}_n)$ (one original motivation)

$\rightarrow \sigma : [n] \to [n]$, permutation.

$$\sigma(G) = \left(\sigma(V), \{\{\sigma(i), \sigma(j)\} : \{i,j\} \in E\}\right)$$

Ex. (A) Check that $\sigma(G(n,p)) \stackrel{d}{=} G(n,p)$ i.e.,

$$\mathbb{P}(G(n,p) = G) = \mathbb{P}(\sigma(G(n,p)) = G) \quad \forall\, G \in \mathcal{G}_n.$$

$\rightarrow$ $\deg_{G(n,p)}(i) = \sum_{j=1}^{n} X_{ij} = \sum_{j \neq i} X_{ij}$

$$\stackrel{d}{=} \text{Bin}(n-1, p).$$

If $p = \frac{\lambda}{n}$, $\lambda > 0$ then $\deg(i) \xrightarrow{d} \text{Poi}(\lambda)$ i.e.,

$[\lambda > n,\ p = (\frac{\lambda}{n} \wedge 1)]$ $\quad P(\deg(i) = k) \to e^{-\lambda}\frac{\lambda^k}{k!} \quad \forall\, k \geq 0.$

$\boxed{\rightarrow \text{SPARSE REGIME:}}$

$E_n = E(G(n,p))$ – Edge set of $G(n,p)$.

$$|E_n| = \frac{1}{2}\sum_{i,j=1}^{n} X_{i,j} = \sum_{1 \leq i < j \leq n} X_{ij}$$

Ex (A) $\mathbb{E}[|E_n|] = \binom{n}{2} p$. (by lin. of Exp.) $\quad [p := p(n) = p_n]$

1. If $n^2 p(n) \to 0 \Rightarrow \mathbb{P}(|E_n| \geq 1) \to 0$ as $n \to \infty$.
2. If $np(n) \to 0 \Rightarrow \deg(i) \xrightarrow{d} 0 \quad \forall\, i.$
3. If $np(n) \to \lambda \Rightarrow \deg(i) \xrightarrow{d} \text{Poi}(\lambda), \ \forall\, i$
4. If $np(n) \to \infty \Rightarrow P(\deg(i) \geq m) \to 1 \quad \forall\, m \geq 1.$

$\deg(i) \xrightarrow{d} \infty.$

Ex (A) – Assignment question.

So we can get locally-finite graphs only if

$np \to \lambda \in [0, \infty)$.

Further if $\lambda = 0$, the graph becomes trivial.

So $\lambda \in (0, \infty)$ is the interesting case

This is called the sparse regime & $\{\sup_n \mathbb{E}\deg_{G_n}(i) < \infty \ \forall i\}$

we'll assume $p = \frac{\lambda}{n}$ for simplicity

[instead of $np \to \lambda$]

Degrees are "finite" in the sparse regime but what about the graph overall?

$G = ([n], E)$, $[n] = \{1, \cdots, n\}$.

$H = ([k], E(H))$ - Another graph, connected. $k \leq$

$$X(H; G) = \sum_{\substack{F \subseteq G \\ |V(F)| = k}} \mathbb{1}(F \cong H), \quad \boxed{\#\text{ of copies of } H \text{ in } G}$$

(no. of copies of H in G)

- Sum is over all subgraphs of G on k vertices.

More useful representation

$$X(H; G) = \frac{1}{C_H} \sum_{\substack{\tau : [k] \to [n] \\ \text{injective}}} \mathbb{1}[\tau(H) \subseteq G]$$

$$\underset{(\text{Ex.})}{=} \frac{1}{C_H} \sum_{\tau : \cdots} \prod_{(i,j) \in E(H)} \mathbb{1}[\tau(i) \overset{G}{\sim} \tau(j)]$$

C_H - # of automorphisms of H, i.e., isom : $H \to H$

· Let $G_n = G(n,p)$

$$\mathbb{1}[\tau(i) \overset{G_n}{\sim} \tau(j)] = X_{\tau(i),\tau(j)}$$

So $X(H; G_n) = \frac{1}{c_H} \sum^{\neq}_{i_1,\dots,i_k} \prod_{(j,l) \in E(H)} X_{i_j, i_l}$

($\Sigma^{\neq}$ – distinct summands)

By linearity of Expectations,

$$\mathbb{E}[X(H;G_n)] = \frac{1}{c_H} \sum^{\neq}_{i_1,\dots,i_k} \mathbb{E}\Big[\prod_{\dots} X_{i_j,i_l}\Big]$$

$$= \frac{k!}{c_H}\binom{n}{k} p^{e_H} \qquad e_H = |E(H)|$$

[$X_{i,j}$'s are indep. & distinct]

<u>PROPOSITION</u>: Let $G_n = G(n,p)$, $p = \frac{\lambda}{n}$, $\lambda \in (0,\infty)$

H as above. $\mathrm{exc}(H) = e_H - k \geq -1$
conn on k vertices

Then

$$\mathbb{E}[X(H;G_n)] \underset{n\to\infty}{\sim} c_H^{-1} \lambda^{e_H} n^{-\mathrm{exc}(H)}.$$

$$\left[a_n \sim b_n \iff \frac{a_n}{b_n} \to 1\right]$$

<u>Proof</u>: $\mathbb{E}[X(H;G_n)] = \frac{1}{c_H} \frac{n!}{(n-k)!} \left(\frac{\lambda}{n}\right)^{e_H}$

$$\sim \frac{1}{c_H} n^k \left(\frac{\lambda}{n}\right)^{e_H}$$

∎

Remarks :

(1) $exc(H) = -1$ iff H is a tree [Tree is a Connected acyclic graph]

$$\frac{1}{n}\mathbb{E}[X(H;G_n)] \Rightarrow \frac{\lambda^{k-1}}{c_H}$$

(2) $H = C_k$ – k-cycle k vertices $(\mathbb{Z}/k\mathbb{Z})$

$exc(C_k) = 0$

$$\mathbb{E}[X(C_k;G_n)] \longrightarrow \frac{\lambda^k}{2k}$$

(3) H has at least two cycles (i.e., $exc(H) \geq 1$)

$$\Rightarrow \mathbb{E}[X(H;G_n)] \to 0$$

Markov's ineq $\Rightarrow$ $\mathbb{P}(X(H;G_n) \geq 1) \to 0$.

$\Rightarrow$ In the sparse regime, the only subgraphs in G_n are trees & unicycles

Also no. of trees >> no. of unicycles.

What does it mean for random rooted graphs?

$G_n = G(n,p)$. Choose $O_n \in [n]$ uniformly at random & independently of G_n.

$(G_n, O_n) \in \mathcal{G}_*$ – loc. fin, rooted graphs

$$X(H;(G,o)) = \sum_{\substack{F \subseteq G \\ o \in F,\, |V(F)| = k}} \mathbb{1}[F \cong H]$$

$$= \frac{1}{C_H} \sum_{i_1,\dots,i_k}^{\neq} \mathbb{1}[o \in \{i_1,\dots,i_k\}] \prod_{(j,l) \in H} \mathbb{1}[i_j \overset{G}{\sim} i_l]$$

$$\mathbb{E}\, X(H;(G_n,o_n)) = \frac{1}{C_H} \sum_{i_1,\dots,i_k} \mathbb{E}\Big[\mathbb{1}[o_n \in \{i_1,\dots,i_k\}] \prod_{\dots} X_{i_j, i_l}\Big]$$

(o_n indep. of $X_{i,j}$'s & Unif $[n]$)

$\Downarrow$

$$= \frac{k}{C_H\, n} \frac{n!}{(n-k)!} p^{e_H}$$

$$\overset{n\to\infty}{\sim} \frac{k}{n} \frac{1}{C_H} \lambda^{e_H} n^{-\mathrm{exc}(H)}$$

$\Rightarrow$ $\mathbb{E}\, X(H;(G_n,o_n)) \longrightarrow \dfrac{k\lambda^{k-1}}{C_H}$ if H is a tree

$\longrightarrow 0$ $\forall$ all conn. $H \neq$ tree.

So, Markov's inequality

$\Rightarrow \mathbb{P}(o_n \in H \text{ in } (G_n, o_n)) \to 0$ if H has a cycle.

$\Rightarrow$ "From o_n, (G_n, o_n) is a tree"!

$\Rightarrow$ From a "typical" vertex in G_n, G_n "looks locally like a tree".

Next few weeks towards showing that

$$(G_n, o_n) \xrightarrow{LW} GW(\lambda)$$

↳ Galton-Watson tree with Poisson(λ) distribn.

Cycle Counts:

Defn: Total variation distance between 2 prob. measures P & Q on $(S, \mathcal{S})$ is defined as

$$d_{TV}(P, Q) := \sup_{A \in \mathcal{S}} |P(A) - Q(A)|$$

observation: $P(A^c) - Q(A^c) = Q(A) - P(A)$

$$\Rightarrow d_{TV}(P, Q) = \sup_{A \in \mathcal{S}} P(A) - Q(A)$$

S is countable; Supremum is achieved for

$$A = \{ x \in S : P(x) \geq Q(x) \}$$

$$P(A) - Q(A) = \sum_{x \in A} |P(x) - Q(x)|$$

$$\quad \parallel$$

$$Q(A^c) - P(A^c) = \sum_{x \in A^c} |P(x) - Q(x)|$$

$$\Rightarrow d_{TV}(P, Q) = \frac{1}{2} \sum_{x \in S} |P(x) - Q(x)|$$

Notn: If $X \stackrel{d}{=} P$, $Y \stackrel{d}{=} Q$ $d_{TV}(X, Y) = d_{TV}(P, Q)$

T.V. distance dep. on prob. distribn & not on r.v.'s.

If $d_{TV}(P_n, P) \to 0$ then $P_n(x) \to P(x)$ $\forall x \in S$.

[S is countable]

If $S = \mathbb{Z}$; $X_n \stackrel{d}{=} P_n$ & $X \stackrel{d}{=} P$ the above $\Rightarrow$

$$X_n \xrightarrow{d} X.$$

Def: Suppose $\{I_a\}_{a \in \Gamma}$ is a coll^n of r.v. We say $L = (\Gamma, E(L))$ is a DEPENDENCY GRAPH for $\{I_a\}$ if whenever $A, B \subseteq \Gamma$ & $\nexists$ no edges between A & B ($A \cap B = \phi$) then $\{I_a\}_{a \in A}$ & $\{I_b\}_{b \in B}$ are independent.

Eg: $\{I_a\}_{a \in \Gamma}$ are indep.; $E(L) = \phi$ works.

Eg: $\{X_i\}_{i=1}^{\infty}$ indep. r.v.'s.

$$I_i = X_i X_{i+1} \cdots X_{i+k}, \quad i \geq 1$$

$$L = (\mathbb{N}, \{(i,j) : |i-j| < 3k\})$$

(Ex.) Then L is a dep. graph for $(I_i)_{i=1}^{\infty}$

$\rightarrow$ Complete graph is always a dep. graph for finite Γ.

THM [Holst, Barbour & Janson] (see Frieze-Karonski THM 20.12)

$X = \sum_{a \in \Gamma} I_a$, I_a are Ber(p_a) rand. variables

& has a dep. graph $L = (\Gamma, E(L))$. $\lambda = \sum_{a \in \Gamma} p_a$.

$Z_\lambda \stackrel{d}{=} Poi(\lambda)$

Then $$d_{TV}(X, Z_\lambda) \leq \min\{\lambda^{-1}, 1\} \left[\sum_{a \in \Gamma} \sum_{b \in N_a \cup \{a\}} p_a p_b + \sum_{a \in \Gamma} \sum_{b \in N_a} \mathbb{E}[I_a I_b] \right]$$

$N_a = \{b : b \stackrel{L}{\sim} a\}$.

Eg: $H = C_3$. Δ $$X(H; G_n) = \frac{1}{6} \sum_{i,j,k}^{\neq} X_{ij} X_{jk} X_{ik}$$

$$= \sum_{\{i,j,k\}}^{\neq} Y_{ijk}$$

$Y_{ijk} = X_{ij} X_{jk} X_{ik}$; $\Gamma = \{\{i,j,k\} : i \neq j \neq k\}$

Y_{ijk} are Ber(p^3) r.v & not indep.

But if $|\underset{\Gamma}{\overset{\cap}{I}} \cap \underset{\Gamma}{\overset{\cap}{J}}| \leq 1$ then Y_I & Y_J are independent, because they don't share an edge.

$|\Gamma| = \binom{n}{3}$

$\Gamma = \{ I \subseteq [n] : |I| = 3 \}$ – All triangles in K_n.

$E(L) = \{ (I, J) : |I \cap J| \geq 2 \}$ – Pair of triangles that share an edge.

(Ex.) S.T. L is dep. graph for $\{Y_I\}_{I \in \Gamma}$.

$d_{TV}(X(C_3; G_n), Poi(\lambda_n)) \leq T1 + T2$ (B+J Thm)

$\lambda_n = \binom{n}{3} p^3 = \frac{n(n-1)(n-2)p^3}{6}$

$$T2 = \sum_{I \in \Gamma} \sum_{\substack{J \sim I \\ J \neq I}} E[Y_I Y_J]$$

$$= \sum_{I \in \Gamma} \sum_{|J \cap I| = 2} E[Y_I Y_J] \Leftarrow \begin{cases} J \sim I, & |J \cap I| \geq 2 \\ J \neq I, & |J \cap I| < 3 \end{cases}$$

Suppose $I = \{i, j, k\}$ & $J = \{i, j, l\}$

the $Y_I Y_J = X_{ij} X_{ik} X_{jk} X_{il} X_{jl}$

Since all are independent

$\Rightarrow E Y_I Y_J = p^5$; $\#\{J : |J \cap I| = 2\} = 3(n-3)$.

$$\Rightarrow T2 = 3(n-3) \binom{n}{3} p^5$$

$$T1 = \sum_{I \in \Gamma} \left(\sum_{\substack{J \sim I \\ J \neq I}} E[Y_I] E[Y_J] + E[Y_I]^2 \right)$$

$$= \binom{n}{3} p^6 \left[3(n-3) + 1 \right]$$

$$\Rightarrow d_{TV}\left(X(C_3, G_n), \mathrm{Poi}\left(\binom{n}{3} p^3\right) \right) \leq 3(n-3)\binom{n}{3}(p^5 + p^6) + \binom{n}{3} p^6 \quad \text{—} (1)$$

THM: Suppose $p = \frac{\lambda}{n}$, $\lambda \in (0, \infty)$.

$$d_{TV}\left(X(C_3, G_n), \mathrm{Poi}\left(\frac{\lambda^3}{6}\right) \right) \longrightarrow 0 \text{ as } n \to \infty.$$

Proof: $d_{TV}(\downarrow \qquad \downarrow)$ By (1)

(Triangle inequality)
$$\leq d_{TV}\left(X(C_3; G_n), \mathrm{Poi}\left(\binom{n}{3} p^3\right) \right) \longrightarrow 0$$

$$+ \, d_{TV}\left(\mathrm{Poi}\left(\binom{n}{3} p^3\right), \mathrm{Poi}\left(\frac{\lambda^3}{6}\right) \right) \longrightarrow 0$$

$$d_{TV}(\mathrm{Poi}(a), \mathrm{Poi}(b)) \leq |a - b|.$$ (Proof next).

DEFN: Given prob. measures P, Q on $(S_1, \mathcal{S}_1)$ & $(S_2, \mathcal{S}_2)$ respectively, a coupling of

P & Q is a prob. measure Π on $(S_1 \times S_2, \mathcal{S}_1 \times \mathcal{S}_2)$
$\ni \quad \Pi \circ \pi_1^{-1} = P \qquad \Pi \circ \pi_2^{-1} = Q$
$(\pi_i : S \to S_i$ projn.$)$

$\boxed{\text{Eg. } \Pi = P \times Q}$

Probabilistically, coupling of random elements $X \stackrel{d}{=} P$ & $Y \stackrel{d}{=} Q$
is a random vector $(\hat{X}, \hat{Y}) \in S_1 \times S_2 \ni$
$\hat{X} \stackrel{d}{=} X$ & $\hat{Y} \stackrel{d}{=} Y$.

Originally $\quad X : (\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \to (S_1, \mathcal{S}_1)$
$\qquad\qquad\quad Y : (\Omega_2, \cdot, \mathbb{P}_1) \to (\quad)$

But $\quad (\hat{X}, \hat{Y}) : (\Omega, \mathcal{F}, \mathbb{P}) \to (\underset{S_1 \times S_2}{S}, \mathcal{S})$

We say $(\hat{X}, \hat{Y})$ is a coupling of X & Y or P & Q.

PROPN: $\quad \underset{S_1 = S_2}{} \; d_{TV}(P, Q) \le \mathbb{P}(\hat{X} \neq \hat{Y})$ for any coupling $(\hat{X}, \hat{Y})$

Proof: $\quad P(A) - Q(A) = \mathbb{P}(\hat{X} \in A) - \mathbb{P}(\hat{Y} \in A)$
$= \mathbb{E}[\mathbb{1}[\hat{X} \in A] - \mathbb{1}[\hat{Y} \in A]] \quad$ [by linearity & coupling]

$$= \mathbb{E}\left[\left(\mathbb{1}[\hat{X} \in A] - \mathbb{1}[\hat{Y} \in A]\right) \mathbb{1}[\hat{X} \neq \hat{Y}]\right]$$

$$\leq \mathbb{P}(\hat{X} \neq \hat{Y}) .$$

$a<b$.

Let Z_a be $\mathrm{Poi}(a)$ r.v., Z_b be $\mathrm{Poi}(b)$ r.v. $a<b$.

& let Y be $\mathrm{Poi}(b-a)$ r.v. & independent of Z_a.

$\Rightarrow$ $(Z_a, Y) : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \cdot)$ exists

Then $\hat{Z}_b = Z_a + Y \overset{d}{=} Z_b$; $\hat{Z}_a = Z_a$.

Prodon

$\Rightarrow d_{TV}(\mathrm{Poi}(a), \mathrm{Poi}(b)) \leq \mathbb{P}(\hat{Z}_b \neq \hat{Z}_a)$

[$(\hat{Z}_b, \hat{Z}_a)$ is a coupling of Z_a & Z_b]

$$= \mathbb{P}(Y \geq 1) = 1 - e^{-(b-a)}$$

$$\leq (b-a)$$

$\Rightarrow d_{TV}(\mathrm{Poi}(a), \mathrm{Poi}(b)) \leq |b-a| \quad \forall a, b.$

$\to$ Completes proof of Poisson approximation THM for $X(C_3; G_n)$.

Remarks:

(1) $\forall\ (P, Q)\ \exists$ a coupling $(\hat{X}, \hat{Y}) \ni$

$$d_{TV}(P, Q) = \mathbb{P}(\hat{X} \neq \hat{Y})$$

[vdH-1 (Ch. 2) ; Bordenave (Ch. 2)].

(2) Poisson Convergence: $X_n \xrightarrow{d} \mathrm{Poi}(\lambda)$

(3) Poisson Approximation: $d_*(X_n, \mathrm{Poi}(\lambda)) \leq \ldots \to 0$.

d_* is a metric on the space of prob. measures.

(4) Suppose $\{X_n\}_{n \geq 1}$ are $\mathbb{N} \cup \{0\}$ - r.v. & $X \stackrel{d}{=} \mathrm{Poi}(\lambda)$.

If $\mathbb{E}[X_n(X_n-1)\cdots(X_n-k+1)] \longrightarrow \lambda^k \quad \forall k \geq 1$

(Factorial moments) $\qquad$ k^{th} Factorial mom. of $\mathrm{Poi}(\lambda)$

then $X_n \xrightarrow{d} \mathrm{Poi}(\lambda)$.

[vdH-1, Ch. 2]

Earlier proofs of $X(C_k; G_n) \xrightarrow{d} \mathrm{Poi}\left(\frac{\lambda^k}{2k}\right)$

follows by factorial moment / moment method.

<u>Eg</u>: $X = \sum_{a \in \Gamma} Y_a$.

$$X(X-1)\cdots(X-k+1) = \sum^{\neq}_{a_1,\dots,a_k \in \Gamma} Y_{a_1}\cdots Y_{a_k}.$$

[Poisson Convergence: Alon & Spencer - Probabilistic Method].

(5) We have rate of convergence of
$d_{TV}(X(C_3; G_n), Z_{\frac{\lambda^3}{6}})$

H conn. subgraph

$exc(H) \geq 1 \Rightarrow X(H; G_n) \to 0$

$H = C_k \Rightarrow X(C_k; G_n) \xrightarrow{d} Poi(\frac{\lambda^k}{2k})$ [Ex (A)]

$exc(H) = -1$ i.e., H is a tree

$n^{-1} X(H; G_n) \xrightarrow{d} ?$

$n^{-1} \mathbb{E} X(H; G_n) \to$ ·

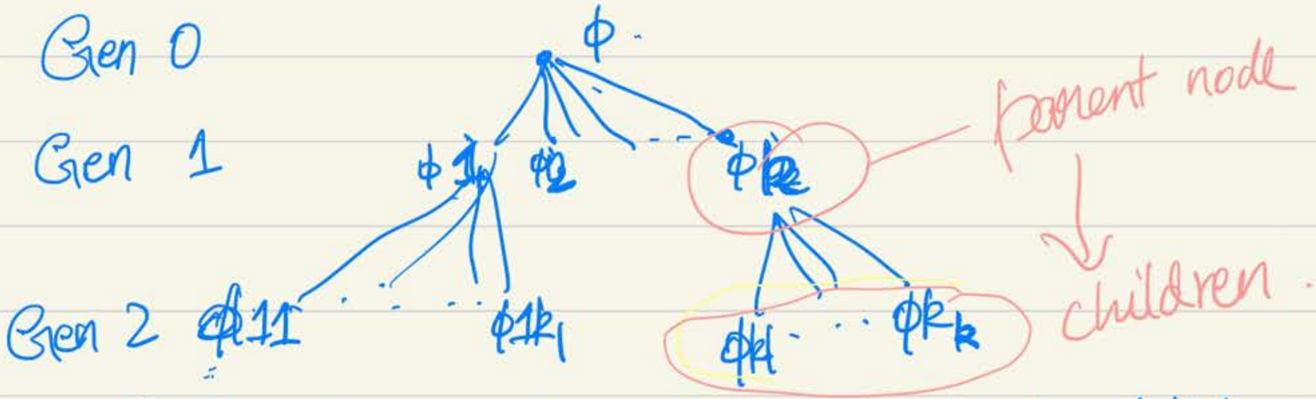
10/9.

L3 - Bienaymé-Galton-Watson Trees.

$\mathbb{N}^f$ — Set of possible tree nodes.

$= \{ (\phi i_1 \cdots i_k) : k \geq 0,\ i_j \geq 1 \}$

ϕ - root. $\quad v = (\phi i_1 \cdots i_k)$ is a vertex in generation k. , $k = |v|$.



$\{ (\phi i_1 \cdots i_{k+1}) : i_{k+1} \geq 1 \}$ are called children of of the node $(\phi i_1 \cdots i_k)$

A sequence $\{ n_v \geq 0 : v \in \mathbb{N}^f \}$ specifies a rooted tree & vice-versa.

BGW

Defn: BGW is a random tree in which each node has i.i.d. offspring distributed as N.
i.e., The offspring are specified by $\{N_v : v \in \mathbb{N}^f\} \ni N_v$ are i.i.d. with the same distribution as N.

Notn: $p_k = \mathbb{P}(N=k)$; $m = \mathbb{E}[N] = \sum k p_k$

$$\phi_N(s) = \phi(s) = \mathbb{E}[s^N] = \sum_{k=0}^{\infty} p_k s^k \qquad s \in [0,1].$$

$$BGW = \{v \in \mathbb{N}^f : v = \phi i_1 \cdots i_k,\ i_1 \le N_\phi,\ i_2 \le N_{\phi i_1},\ \ldots,\ i_k \le N_{\phi i_1 \cdots i_{k-1}}\}$$

Eg: $N_\phi = 1$, $N_{\phi 1} = 0$

$$X_n = \#\{v \in BGW \mid |v| = n\}$$ — no. of individuals in n^{th} gen
vertices

$X_1 = 1, \quad X_2 = N_{(1)}.$

$$X_n = \sum_{\substack{v \in BGW \\ |v| = n-1}} N_v \overset{d}{=} \sum_{i=1}^{X_{n-1}} N_i$$

N_i i.i.d as N.

$$S = \sum_{n=1}^{\infty} X_n$$ — Total no. of individuals in BGW tree.

$X_n > 0 \iff X_n \geq 1.$

$X_n = 0 \implies X_{n+1} = 0.$

$\implies S < \infty \iff X_n = 0$ for some n

Extinction $= \{S < \infty\} = \{X_n = 0 \text{ for some } n\}$

$= \{\text{BGW is a finite tree}\}$

$p_{ext} = \mathbb{P}(S < \infty) = \mathbb{P}(\text{BGW is finite}).$

THM: p_{ext} is the smallest soln in s of the eqn $\underset{\underset{\mathbb{E}[s^N]}{\shortparallel}}{\phi(s)} = s \quad \forall s \in [0,1]$

Proof: $\{X_n = 0\} \uparrow$

$$\Rightarrow p_{ext} = \mathbb{P}\Big(\bigcup_{n\geq 1} \{X_n = 0\}\Big) = \lim_{n\to\infty} \mathbb{P}(X_n = 0)$$

$$= \lim_{n\to\infty} \phi_{n+1}(0) = \lim_{n\to\infty} \phi_n(0)$$

$\phi_{n+1}(s) = \mathbb{E}[s^{X_{n+1}}]$.

$$\phi_n(s) = \mathbb{E}[s^{X_n}] = \sum_{k=0}^{\infty} \mathbb{P}(X_{n-1} = k)\, \mathbb{E}\Big[s^{\sum_{i=1}^{k} N_i} \,\Big|\, X_{n-1} = k\Big]$$

(N_i's are indep of X_{n-1})

$$= \sum_{k=0}^{\infty} \mathbb{P}(X_{n-1} = k)\, \phi(s)^k$$

$$= \mathbb{E}\big[\phi(s)^{X_{n-1}}\big] = \phi_{n-1}(\phi(s))$$

induction

$$= \phi^n(s) = \phi(\phi_{n-1}(s))$$

$$p_{ext} = \lim_{n\to\infty} \phi_n(0) = \phi\left(\lim_{n\to\infty} \phi_{n-1}(0)\right) = \phi(p_{ext}).$$

[by cty of ϕ]

$\Rightarrow p_{ext}$ is a sol^n of $p_{ext} = \phi(p_{ext})$.

If $\eta \in [0,1]$ is another of $\eta = \phi(\eta)$.

$0 \le \eta$

ϕ is $\uparrow \Rightarrow \phi(0) \le \phi(\eta) = \eta$

$\Rightarrow \phi^n(0) \le \eta \Rightarrow p_{ext} \le \eta$. ■

COR: (1) If $m < 1$ then $p_{ext} = 1$ (sub-critical)

(2) If $m > 1$ then $p_{ext} < 1$ (super-critical)

$\Rightarrow P(S=\infty) > 0$

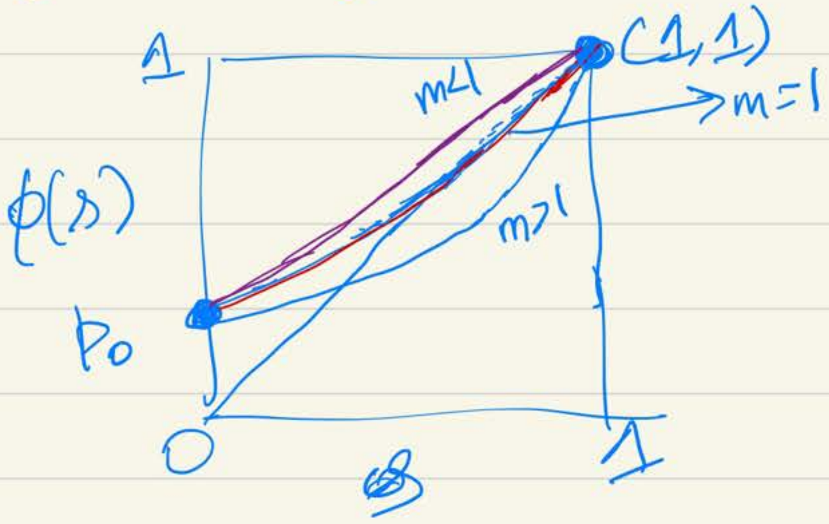
(3) $m = 1$ (critical)

$\rightarrow p_0 = 0$ ($\Leftrightarrow p_1 = 1$) then $p_{ext} = 0$

$\rightarrow p_0 > 0$ the $p_{ext} = 1$.

[$p_{ext} \ge p_0$. $p_0 = 0 \Rightarrow p_{ext} = 0$.]

Proof: $\phi(0) = p_0$, $\phi(1) = 1$.



$\rightarrow$ ϕ is convex. $m = \phi'(1) > 0$.

$\Rightarrow$ ϕ has atmost one soln < 1.

& if the sol^n exists then it is p_{ext}.

$\Leftrightarrow$ $m < 1$ or $m = 1$ & $p_0 > 0$.

[Ch. 2 ; B. Blaszczyszyn -] .

N^f - has natural order. $u < v$ if $|u| < |v|$

$|u| = |v|$, then order by lexicographic ordering on indices.

(Breadth first order)

For eg. $\phi 112 < \phi 113 < \phi 121$

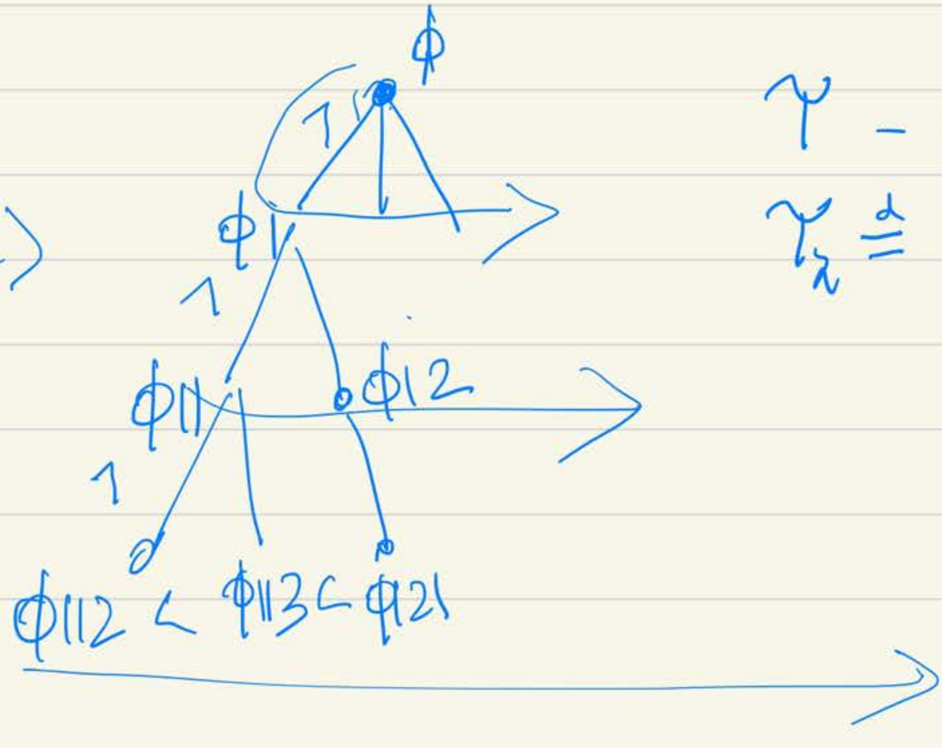
Let $(D_i)_{i=1}^{n}$ be the no. of offsprings of the "first n" vertices in BGW.

$D_1 = N_\phi$.

LEMMA: $\mathbb{P}(D_i = d_i, 1 \le i \le n) = \prod_{i=1}^{n} P_{d_i}$ provided (d_i) is a possible realization

Notn: BGW(λ) — BGW where $N \stackrel{d}{=} \mathrm{Poi}(\lambda)$

$\rightarrow \mathbb{P}(D_i = d_i, 1 \le i \le n) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{d_i}}{d_i!}$ ∎



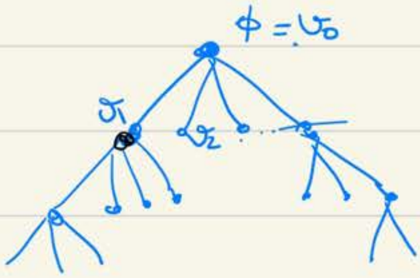
Υ - tree

$\Upsilon_\lambda \stackrel{d}{=}$ BGW(λ)

22/09/20 : L4 - Breadth-first exploration of Random graphs

Exploration of BGW tree

$\mathcal{T}$ - tree



— Breadth-first ordering

$\mathcal{A}_k$ - Active nodes at step k ; ξ_k - no. of newly discovered nodes.

$A_k = |\mathcal{A}_k|$

black - active
red - deactivated.

$k = 0$; $\mathcal{A}_0 = \{\phi\}$; $A_0 = 1$.

$k = 1$; $\mathcal{A}_1 = N(v_0) = \mathcal{A}_0 \cup N_{v_0} \setminus \{v_0\}$

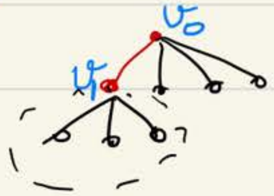
i.e, discover new nbhrs of v_0 & deactivate v_0.

$\xi_1 = |N(v_0)|$ & $A_1 = A_0 + \xi_1 - 1 = \xi_1$.

$k = 2$; Choose smallest node in $\mathcal{A}_1$ - say v_1.

Deactivate v_1 & discover all new nbhrs of v_1. $\leftrightarrow$ children [because $\mathcal{T}$ is a tree]

$\xi_2 = |N(v_1)| - 1$. $A_2 = A_1 + \xi_2 - 1$



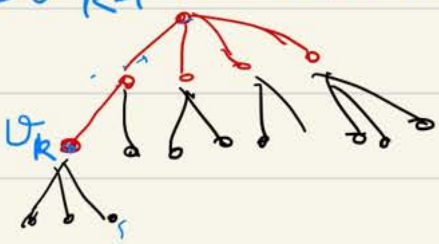
$N(v)$ - neighbourhood of v.

k − Choose smallest node in $\mathcal{A}_{k-1}$ say v_k.

Deactivate v_k & discover all children of v_k.

ξ_k = # children of v_k = $|N(v_k)| - 1$

$A_k = A_{k-1} + \xi_k - 1$; $\mathcal{A}_k = \mathcal{A}_{k-1} \cup N(v_k) \setminus \{v_k\}$



Exploration can continue only as long as there are active nodes

i.e., $T = \min\{k \geq 1 : A_k = 0\}$. $\min \phi = \infty$.

$T < \infty$ iff $\mathcal{T}$ is finite.

Even more, $T = |\mathcal{T}|$, at each time step one node is de-activated.

Total de-activated nodes = Total discovered nodes

$$T = \sum_{k=1}^{T} \xi_k + 1.$$ [Easy for $T < \infty$ & holds for $T = \infty$] as well.

Further if $\mathcal{T} \stackrel{d}{=} BGW(\underbrace{N}_{\text{offspring distribn}})$ then ξ_k are iid. with distribn N. T & ξ_k's aren't independent!

So far the ordering of nodes do not matter!!!

Breadth-first is a convenient choice as it explores earlier generations | nbhrs of the root first.

i.e., if $d_{\mathcal{T}}(v,\phi) < d_{\mathcal{T}}(w,\phi)$ then v becomes active before w.

- $\mathcal{H} = (\xi_1, \ldots, \xi_T)$ - History of the tree $\mathcal{T}^w$.

$\exists$ Bijn between $\underline{\underline{\mathcal{H}}}$ & $\mathcal{T}$. [check]

Also, $(x_1, \ldots, x_k)$ with $x_i \geq 0$, $1 \leq k \leq \infty$ is a (valid) realization of history $\mathcal{H}$ iff

$$\sum_{i=1}^{l} x_i + 1 > l \quad \forall\, l = 1, \ldots, k-1$$ (if $\exists$ active nodes, more discovered than deactivated)

$$\sum_{i=1}^{k} x_i + 1 = k.$$ (if $\exists$ no active nodes discovered = deactivated).

[Blaszczyszyn, Ch. 2]

$$\mathbb{P}(\mathcal{H} = (x_1, \ldots, x_k)) = \prod_{i=1}^{k} p_{x_i} \quad \text{for } \mathcal{T} \overset{d}{=} BGW(N)$$

(ξ_i's are i.i.d.) $\quad p_k = \mathbb{P}(N=k)$.

→ Very useful tool - Encodes a random tree as a sequence of random variables.

Or as a random walk with step distribution $\xi_i - 1$.

[Abraham & Delmas - notes on BGW]

Easy Applns:

$$\underset{(N)}{\text{Super-critical BGW}} \mid \{\text{extinction}\} \overset{d}{=} \underset{(\hat{N})}{\text{Sub-critical BGW}}.$$

Thm 2.2.5 of Blaszczyszyn.

$G=(V,E)$.

EXPLORATION OF A GRAPH. Fix an ordering.

Let $v_0 = v$.

$k=0$. $A_0 = \{v\}$, $B_0 = \phi$ (discovered but inactive i.e., deactivated)

$k=1$, Pick $v_1 = v$. deactivate it.

Set $D_1 = D(v_1) = N(v_1)$ — newly discovered nodes.

$A_1 = A_0 \cup D_1 \setminus \{v_1\}$.

$B_1 = \{v_1\}$.

$\xi_1 = |D_1|$ - label D_1 as $v_2, \ldots, v_{\xi_1}$ as per initial order.

Step k: If $A_{k-1} \neq \phi$, pick $v_k \in A_{k-1}$ as per initial order. Deactivate v_k.

Set $D_k = D(v_k) = N(v_k) \setminus (A_{k-1} \cup B_{k-1})$.

$A_k = A_{k-1} \cup D_k \setminus \{v_k\}$

$B_k = B_{k-1} \cup \{v_k\}$.

$A_k = |A_k|$, $B_k = |B_k|$, $\xi_k = |D_k|$

$\Rightarrow$ $A_0 = 1$; $B_0 = 0$ . . . $B_k = k$.

$A_k = A_{k-1} + \xi_k - 1$.

Suppose G is a graph on n vertices then knowing $\xi_1, \ldots, \xi_k$ restricts choices for ξ_{k+1}

[For eg. $\xi_{k+1} \leq n-1-\sum_{i=1}^{k} \xi_k$]

So if $G_n = G(n,p)$, $\xi_1, \dots, \xi_k$ aren't independent.

Also ξ_i needn't be equal to $|N(v_i)|$! (not always).

Recall from assignment 1, $\forall m \geq 1$

$$\mathbb{P}(D_1 = k_1, \dots, D_m = k_m) \to \boxed{\prod_{i=1}^{m} e^{-\lambda} \frac{\lambda^{k_i}}{k_i!}} \quad \left(p = \frac{\lambda}{n}\right)$$

$D_i = \deg(i)$

i.e., if we pick m arbitrary vertices & look at their degrees they are indep. $\mathrm{Poi}(\lambda)$ r.v.'s !

But in a BGW(*), even if we look at the first 'm' nodes this is true!

We'll show this for $G(n,p)$, $p = \frac{\lambda}{n}$ now.

From Breadth-First exploration of $G_n = G(n,p_n)$, $p_n = \frac{\lambda}{n}$

$$A_k = A_{k-1} + \xi_k - 1 \;, \quad B_k = k.$$

- ξ_k = # new neighbours of v_k

$$= \Big| \{ v' \in [n] : X_{v_k, v'} = 1 \;\&\; v' \notin A_{k-1} \cup B_{k-1} \} \Big|$$

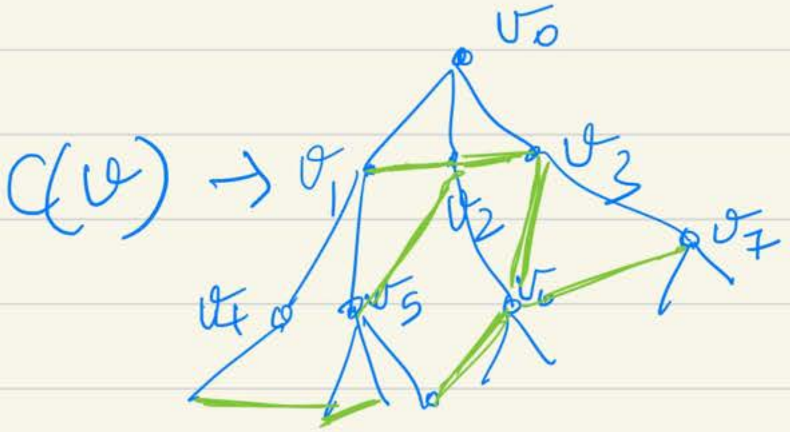
$$= \sum_{v \notin A_{k-1} \cup B_{k-1}} X_{v_{k-1}, v}$$

$$\stackrel{d}{=} \mathrm{Bin}\left(n - |A_{k-1}| - |B_{k-1}|, p_n\right) \quad (X_{i,j}\text{'s are i.i.d } \mathrm{Ber}(p))$$

$$\stackrel{d}{=} \mathrm{Bin}\left(n - \sum_{i=1}^{k-1} \xi_i - 1, p_n\right).$$

$$T = \min\{k : A_k = 0\} = |C(v)|$$

$C(v)$ – component of v in G_n



→ undiscovered edges

edges from v_k to $A_{k-1} \cup B_{k-1}$.

BF exploration of a Graph starting at v, yields a spanning tree of $C(v)$.

$$\mathcal{H}_n = \mathcal{H}_n(v) = (\xi_1, \ldots, \xi_T) \text{ – History of nbhd of } v.$$

$(x_1, \dots, x_k)$ is a possible realization of $\mathcal{H}_n$ upto k steps iff

These two condns are same as for Trees
$$\begin{cases} \sum_{i=1}^{j} x_i - j + 1 > 0 & 1 \le j < k \\ \sum_{i=1}^{k} x_i - k + 1 \ge 0 & \text{with } = \text{ iff } k = T. \end{cases}$$

Total no. of vertices $\le n$.
$$\sum_{i=1}^{k} x_i + 1 \le n.$$

THM: Let $\mathcal{H}_n = \mathcal{H}_n(v)$ be history of v in G_n as given above. Let $x = (x_1, \dots, x_k)$ be a possible realization of $\mathcal{H}_n$ upto step k. Then

(a)
$$\lim_{n \to \infty} \mathbb{P}(\xi_1 = x_1, \dots, \xi_k = x_k)$$

[Ch. 3 of Blaczszyczyn (BB)]

$$= \prod_{i=1}^{k} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!}$$

$$= \mathbb{P}(\xi_i' = x_i, 1 \le i \le k)$$

$\mathcal{H}_\lambda = (\xi_1', \dots, \xi_{T}')$ - History of T_λ - BGW(λ).

(b). $\zeta_{i} = |N(v_i)| - \xi_i - 1, \ 1 \le i \le k.$

$$\mathbb{P}\left((\xi_i, \zeta_i) = (x_i, y_i), 1 \le i \le k \right)$$

$$\longrightarrow \begin{cases} \prod_{i=1}^{k} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} & \text{if } y_1 = -1, \ y_2 = \cdots = y_k = 0. \\ 0 & \text{else.} \end{cases}$$

~~Proof~~: (a) Given x_i's, define $a_i = \sum_{j=1}^{i} x_j - i + 1$

$b_i = i, \ 1 \le i \le k. \ ; \ a_i + b_i = \sum_{j=1}^{i} x_j + 1.$

By defn of ξ_j

$$\mathbb{P}(\xi_j = x_j \mid \xi_1 = x_1, \ldots, \xi_{i-1} = x_{i-1})$$

$$= \mathbb{P}\left(\mathrm{Bin}\left(n - \sum_{i=1}^{j-1} x_i - 1, \frac{\lambda}{n}\right) = x_j \right)$$

$$= \binom{n - a_{j-1} - b_{j-1}}{x_j} p_n^{x_j} (1 - p_n)^{n - a_{j-1} - b_{j-1} - x_j}$$

$p_n = \frac{\lambda}{n}.$

$$\mathbb{P}(\xi_j = x_j;\ 1 \le j \le k)$$

$$= \mathbb{P}(\xi_1 = x_1) \prod_{j=2}^{k} \mathbb{P}(\xi_j = x_j \mid \xi_1 = x_1, \dots, \xi_{j-1} = x_{j-1})$$

(by successive conditioning)

$$\text{(above computation)} = \prod_{j=1}^{k} \binom{n - a_{j-1} - b_{j-1}}{x_j} p_n^{x_j} (1-p_n)^{n - a_{j-1} - b_{j-1} - x_j}$$

$$\longrightarrow \prod_{j=1}^{k} \frac{\lambda^{x_j}}{x_j!} \times e^{-\lambda}$$

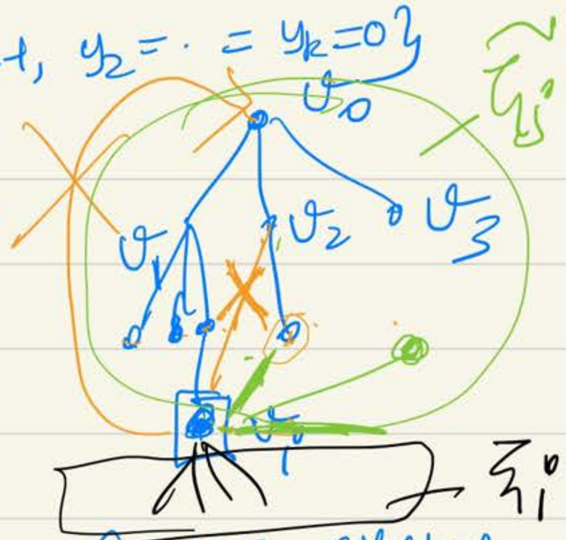
$$= \prod_{j=1}^{k} \frac{\lambda^{x_j}}{x_j!} e^{-\lambda}.$$ Proves (a)

(b) $\tau_1 = t\ ;\ \tau_i \overset{d}{=} \mathrm{Bin}(|A_{i-1}| - 1, p_n)$

$2 \le i \le k.$

$$\le \underline{\tilde{\tau}_i} \overset{d}{=} \mathrm{Bin}(|A_{i-1}| + |B_{i-1}| - 1, p_n)$$

suppose $(y_1, \ldots, y_k) \notin \{ y : y_1 = -1, y_2 = \cdots = y_k = 0 \}$



$$\mathbb{P}\big((\xi_i, \zeta_i) = (x_i, y_i), 1 \le i \le k \big)$$

$$\le \mathbb{P}\big(\xi_i = x_i, 1 \le i \le k, \ \tilde{\zeta}_i \ge y_i, 1 \le i \le k \big)$$

$$= \prod_{j=1}^{k} \binom{n - a_{j-1} - b_{j-1}}{x_j} p_n^{x_j} (1-p_n)^{n - a_{j-1} - b_{j-1} - x_j} \boxed{\mathbb{P}(\tilde{\zeta}_j \ge y_j)}$$

[WLOG assume $y_1 = -1$]

$\Downarrow$

$\exists \ y_j > 0$.

[ξ_j & $\tilde{\zeta}_j$ are independent given $A_{k-1} \cup B_{k-1}$]

[again use successive conditioning]

$\tilde{\zeta}_j = \text{Bin}(\underline{a_{j-1} + b_{j-1} - 1}, p_n)$ & $p_n \to 0$

$\Rightarrow \underline{\mathbb{P}(\tilde{\zeta}_j \ge y_j)} \longrightarrow 0$ if $y_j \ge 1$.

[check]

$\Rightarrow \mathbb{P}((\zeta_i, \ell_i) = (x_i, y_i),\ 1 \le i \le k) \to 0$

unless $y_1 = -1,\ y_2 = \cdots = y_k = 0$.

Suppose $y_1 = -1,\ y_2 = \quad = y_k = 0$.

$$\left.\begin{array}{l} \mathbb{P}(\zeta_i \ge 0) = 1. \\ \mathbb{P}(\tilde{\zeta}_i = 0 \mid A_{i-1} = a_{i-1}, B_{i-1} = b_{i-1}) \to 1 \\ \mathbb{P}(0 \le \zeta_i \le \tilde{\zeta}_i) = 1 \end{array}\right\}$$

$$\mathbb{P}(\zeta_i = 0 \mid A_{i-1} = a_{i-1}, B_{i-1} = b_{i-1}) \to 1.$$

$\Rightarrow$ Repeating same arguments as above we obtain part (b) of the theorem. ∎

[Hidden in proof of THM 2.11 of Hofstad V2]

29/9/20: L9 - Local-tree structure of ER Random graphs.

<u>THM</u>. Consider $G_n = G(n, p_n)$, $p_n = \frac{\lambda}{n}$, $\lambda \in (0,\infty)$.

Let T be a finite rooted tree.

$$\mathbb{P}\left(B_r^{(G_n)}(1) \cong T\right) \to \mathbb{P}\left(B_r^{(\mathcal{T}_\lambda)}(\phi) \cong T\right)$$

where $\mathcal{T}_\lambda \overset{d}{=} BGW(\lambda)$.

Proof: Let T be a rooted tree of at most r generations.

Let $\mathcal{H}_T = (x_1, \dots, x_m)$ - History of BF exploration of T.

Let $\mathcal{H}_{\tau}^{m} = (\xi_1^1, \dots, \xi_m^1)$ be history of $\tau = \tau_\lambda$ upto m^{th} step. [Since T has history upto m]

Let $T_1, \dots, T_\ell$ be all rooted isomorphic copies of T $\ni$

$\mathcal{H}_{T_1}, \dots, \mathcal{H}_{T_\ell}$ are all distinct. Of course

$\mathcal{H}_{T_i} = (x_{\sigma^i(1)}, \dots, x_{\sigma^i(m)})$ for $\sigma^i : [m] \to [m]$ bijn.

$$\{B_r^{(\tau)}(\phi) \cong T\} = \bigsqcup_{i=1}^{l} \{\mathcal{H}_\tau^m = \mathcal{H}_{T_i}\}$$

as H_{T_i}'s are distinct the union is disjt union.

$$\Rightarrow \mathbb{P}(B_r^{(\tau)}(\phi) \cong T) = \sum_{i=1}^{l} \mathbb{P}(\mathcal{H}_\tau^m = \mathcal{H}_{T_i})$$

$$= \sum_{i=1}^{l} \prod_{j=1}^{m} e^{-\lambda} \frac{\lambda^{x_j}}{x_j!}.$$

$$\{B_r^{(G_n)}(1) \cong T\} = \bigcup_{i=1}^{l} \{\mathcal{H}_n^m = \mathcal{H}_{T_i}\} \cap \{\tau_{j} = 0 \ 2 \le j \le m, \ \tau_1 = 1\}$$

$\mathcal{H}_n^m = (\xi_1, \dots, \xi_m)$. - History of $C(1)$ in G_n.

$$\text{So } \mathbb{P}(B_r^{(G_n)}(1) \cong T) = \sum_{i=1}^{l} \mathbb{P}(\mathcal{H}_n^m = \mathcal{H}_{T_i}, \ \tau_1 = 1, \ \tau_j = 0, \ 2 \le j \le m)$$

$$= \sum_{i=1}^{l} \mathbb{P}((\xi_j, \tau_j) = (x_{\sigma^i(j)}, 0), \ 2 \le j \le m, \ (\xi_1, \tau_1) = (x_{\sigma^i(1)}, 1))$$

(By prev. thm) $\longrightarrow \displaystyle\sum_{i=1}^{l} \prod_{j=1}^{m} e^{-\lambda} \frac{\lambda^{x_j}}{x_j!} = \mathbb{P}(B_r^{(\tau)}(\phi) \cong T)$.

∎

REMARKS:

(1) ER graph from any vertex is asymptotically $\mathrm{BGW}(\lambda)$ tree-like!

(2) $\boxed{\begin{matrix}\text{Ex}\\ \text{(A)}\end{matrix}}$ $\mathbb{P}(|C^{(G_n)}(1)| \leq k) \to \mathbb{P}(|\mathcal{T}_\lambda| \leq k)$

$$\Rightarrow \lim_{k\to\infty} \lim_{n\to\infty} \mathbb{P}(|C^{(G_n)}(v)| \leq k) = \mathbb{P}(|\mathcal{T}_\lambda| < \infty)$$

$$= P_{ext} \quad \text{(extinction prob.)}$$

$$= \begin{cases} 1 & \text{if } \lambda \leq 1 \\ < 1 & \text{if } \lambda > 1 \end{cases}$$

(3) $\boxed{\begin{matrix}\text{Ex}\\ \text{(A)}\end{matrix}}$ $\forall k \geq 1$ & $n \geq 1$

$$\mathbb{P}(|C^{G_n}(1)| > k) \leq C_\lambda e^{-C_\lambda k}$$

if $\lambda < 1$.

[Asymptotic version of (3) follows from (2)]

(4) $|C_n^{max}| := \max_{i=1,\dots n} |C_n^{(G_n)}(i)|$, $\lambda < 1$

$\mathbb{P}(|C_n^{max}| \geq \alpha_\lambda \log n) \to 0$ for some $\alpha_\lambda > 0$.

(follows from (3) & union bd).

See Exercises in CH. 3 of BB for
more finer properties.

Local limits of BGW trees — R. Abraham
& J-F. Delmas.

Sparse Random Graphs : Assignment 1

Yogeshwaran D.

September 11, 2020

Submit solutions via Moodle by 20th September 10:00 PM.

1. Are the following functionals continuous ?

   (a) $h : \mathcal{G}_* \to \{0,1\}$ is defined as $h((G,o)) = 1[B_r^G(o) \cong H]$, where $r \geq 0$ and H is a connected graph.

   (b) $h : \mathcal{G}_* \to \{0,1\}$ is defined as $h((G,o)) = 1[|\partial B_1^G(o)| = l_1, \ldots, |\partial B_m^G(o)| = l_m]$ where $\partial B_r^G(o) = B_r^G(o) \setminus B_{r-1}^G(o)$ and $l_1, \ldots, l_m \in \{0,1,2,\ldots\}^m$.

   (c) $h : \mathcal{G}_* \to \{0,1\}$ is defined as $h((G,o)) = 1[|C(o)| \geq k]$ where $C(o)$ is the component of origin.

2. Let $D_i, i \in [n]$ be the degrees of the vertices in $G(n,p)$ for $p = \lambda/n, \lambda \in (0,\infty)$. Show that for any $m \geq 1$,

$$\lim_{n\to\infty} \mathbb{P}(D_i = k_i, 1 \leq i \leq m) = \prod_{i=1}^{m} \frac{e^{-\lambda}\lambda^{k_i}}{k_i!}.$$

3. Let H be a connected graph on k vertices , $k > 1$. For any subgraph $H_1 \subset H$, we denote $\frac{|E(H_1)|}{|V(H_1)|}$ by $d(H_1)$, called *the density*. Further, set $m(H) = \max\{d(H_1) : H_1 \subset H\}$. Show the following

   (a) If $p = o(n^{-1/m(H)})$ then $\mathbb{P}(H \subset G(n,p)) \to 0$.

   (b) If $p = \omega(n^{-1/m(H)})$ then $\mathbb{P}(H \subset G(n,p)) \to 1$.

4. Let T be a finite tree on k vertices. Let $X^*(T,G)$ denote the number of components in G isomorphic to T i.e.,

$$X^*(T,G) := \sum_{F \subset G; |V(F)|=k} 1[F \cong T]1[F \text{ is a component in } G].$$

   Let $G(n,p)$ be the ER random graph with $p = \lambda/n, \lambda \in (0,\infty)$. Show that $n^{-1}\mathbb{E}[X^*(T,G(n,p))]$ converges as $n \to \infty$ and also find the limit.

5. Show that for $p = \lambda/n, \lambda \in (0,\infty)$, we have that [1]

$$X(C_4, G(n,p)) \xrightarrow{d} Poi(\frac{\lambda^4}{8}).$$

[1] Anyone is welcome to try for general C_k.

6. Denote by $\phi_S(t)$ the probability generating function of the total number of nodes in the GW tree; $\phi_S(t) := \mathbb{E}[t^S]$ where S is the number of nodes in the GW tree. Show that

$$\phi_S(t) = t\phi_N(\phi_S(t)), s \in [0,1],$$

where ϕ_N is the probability generating function of the off-spring random variable N.

29/09 L5 - Weak convergence & LWC.

Weak Convergence: [Ref: Sec 3.1 of CB].

DEF: $X, X_n, n \geq 1$ sequence of random variables
$X_n \xrightarrow{d} X$ (convergence in distribution or weak convergence)
if $F_{X_n}(x) \longrightarrow F_X(x)$ $\forall x$ where F_X is cts.
($F_X(x) = \mathbb{P}(X \leq x)$, CDF of X)

$\rightarrow$ $X_n \xrightarrow{d} X$ iff $\mathbb{E} f(X_n) \longrightarrow \mathbb{E} f(X)$ $\forall$ bdd cts fns.

This motivates a general definition of weak convergence of random elements in a metric space or prob. measures on a metric space.

Recall $\mathbb{E} f(X) = \int f(x) \mathbb{P}_X(dx)$, $\mathbb{P}_X(A) = \mathbb{P}(X \in A)$
prob. distribn of X.

DEF: Let X_n, X be random elements in a Polish space $(S, \mathcal{S})$. Then $X_n \xrightarrow{d} X$ or $\mathbb{P}_{X_n} \xrightarrow{d} \mathbb{P}_X$ if

$$\mathbb{E} f(X_n) \longrightarrow \mathbb{E} f(X) \quad \forall \text{ bdd cts } f.$$

THM (Portemanteau Theorem)

TFAE

(1) $X_n \xrightarrow{d} X$.

(2) $\mathbb{E}f(X_n) \longrightarrow \mathbb{E}f(X)$ $\forall$ bdd, unif. cts fns f.

(3) $\overline{\lim}\, \mathbb{P}(X_n \in F) \le \mathbb{P}(X \in F)$ $\forall$ closed sets F

(4) $\underline{\lim}\, \mathbb{P}(X_n \in G) \ge \mathbb{P}(X \in G)$ $\forall$ open sets G

(5) $\lim \mathbb{P}(X_n \in A) = \mathbb{P}(X \in A)$ $\forall A \in \mathcal{S}$ $\ni$ $\mathbb{P}(X \in \partial A) = 0$.

Rmk: $F_X(x)$ is cts at x iff $\mathbb{P}(X = x) = 0$.

$F_X(x) = \mathbb{P}(X \in [-\infty, x])$

$X_n \xrightarrow{d} X$ = convergence in distribution or weak convergence.

$\mathcal{G}_*$ – space of loc-finite rooted (connected) graphs.

G – graph disconnected & C_o – conn. component of o

$$(G,o) = (C_o,o)$$

$$C_o = \quad V(C_o) = \{u : d_G(o,u) < \infty\}$$

$$E(C_o) = \{(u,v) \in E(G) : u,v \in V(C_o)\}$$

(can be disconn.).

DEFN (LWC): Let G_n be a finite graph. Let $o_n \overset{d}{=} \text{Unif}(V_n)$ with $V_n = V(G_n)$, $E_n = E(G_n)$.

We say $G_n \xRightarrow{LW} (G,o)$ ((G,o) is a random element of $\mathcal{G}_*$ with prob. dist. μ)

[G_n converges to (G,o) in local weak topology]

if $\mathbb{E}_n[\underline{h(G_n,o_n)}] \Rightarrow \mathbb{E}_\mu[h\underline{(G,o)}]$

$\forall$ bdd cts $h : \underline{\mathcal{G}_*} \to \mathbb{R}$.

$$\mathbb{E}_n\left[h(G_n, o_n)\right] = \frac{1}{n}\sum_{u\in[n]} h(G_n, u)$$

$\boxed{\text{Remarks:}}$ (1) Convergence of a graph to a rooted graph

(2) By defn, it means from most vertices, the graph "looks like" (G, o) "locally" for large n.

(3) Even if we consider G_n to be deterministic graphs, it is possible that (G, o) is random.

<u>THM:</u> Let G_n be as above. $G_n \xrightarrow{LW} (G, o)$ iff $\forall\, H_* \in \mathcal{G}_*$ we have that

$$\frac{1}{n}\sum_{u\in[n]} \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_*\right] \to \mathbb{P}\left(B_r^{G}(o) \cong H_*\right)$$

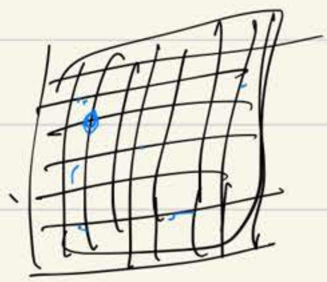
$G_n \xrightarrow{LW} G \implies \forall H_* \in \mathcal{G}_*$

$$\frac{1}{n}\sum_{u\in[n]} \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_*\right] \to \mathbb{P}\left(B_r^{(G)}(o) \cong H\right)$$

[Take $h(G,u) = \mathbb{1}\left[B_r^{(G)}(u) \cong H\right]$

bdd & cts. $\S$ Assignment 1]

Eg: $G_n = \mathbb{Z}^d \cap [-n,n]^d$



$G_n \xrightarrow{LW}$?

$(G_n, 0) \longrightarrow (\mathbb{Z}^d, 0) \quad n \to \infty.$

Fix r: $W_n = [-n,n]^d \quad \underline{W_{n,r}} = \{v \in V_n : d(v, \partial W_n) \le 2r\}$

$$\frac{|V_n \cap W_{n,r}|}{|V_n|} \longrightarrow 1 \quad \text{as } n \to \infty.$$

'cos $|V_n| \sim (2n)^d \quad |W_n \setminus W_{n,r}| \le C_d n^{d-1} r$

. . . . .

$$\Rightarrow \quad \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_*\right] = \mathbb{1}\left[B_r^{\mathbb{Z}^d}(0) \cong H_*\right] \quad \forall u \in W_{n,r}$$

$$\Rightarrow \quad \frac{1}{|V_n|} \sum_{u \in V_n} \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_*\right] = \frac{\mathbb{1}\left[B_r^{\mathbb{Z}^d}(0) \cong H_*\right] |V_n \cap W_{n,r}|}{|V_n|} + \underbrace{\frac{1}{|V_n|} \sum_{u \notin W_{n,r}} \mathbb{1}[\cdots]}_{\to 0}$$

$$\longrightarrow \mathbb{1}\left[B_r^{(\mathbb{Z}^d)}(0) \cong H_*\right].$$

$$\Rightarrow \quad G_n \xrightarrow{LW} (\mathbb{Z}^d, 0).$$

Ex (A) Amenable Cayley Graphs.

Ex. $\mathbb{T}_n^d = \mathbb{Z}^d \cap [0, n]^d / \sim$ (discrete torus)

$= (\mathbb{Z}/n\mathbb{Z})^d \qquad (\mathbb{T}_n^d, u) \cong (\mathbb{T}_n^d, v)$

$$\Rightarrow \quad \mathbb{T}_n^d \xrightarrow{LW} (\mathbb{Z}^d, 0).$$

6/10/20: L6 - LWC FOR RANDOM GRAPHS

THM: Let G_n be as above. $G_n \xrightarrow{LW} (G,o)$ iff $\forall$ $H_* \in \mathcal{G}_*$ we have that

$$\frac{1}{n} \sum_{u \in V_n} \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_* \right] \to \mathbb{P}\left(B_r^{G}(o) \cong H_* \right)$$

Proof: $\Longrightarrow$ as $\mathbb{1}\left[B_r^{(G)}(o) \cong H_* \right]$ is cts. (see A1, Q1)

$\Longleftarrow$ (non-trivial & more useful).

Let $h: \mathcal{G}_* \to \mathbb{R}$ be bdd cts & $\varepsilon > 0$.

By Portmanteau theorem, we can assume that h is uniformly cts as well.

So $\exists$ $\delta > 0$ $\ni$ $d((G_1'', o''), (G_1', o')) < \delta$

$$\Longrightarrow |h(G_1'', o'') - h(G_1', o')| < \varepsilon/4$$

Choose $r = r(\delta)$ $\ni$ $\forall$ (G_1', o')

$$d((G_1', o'), B_r(o')) \leq \frac{1}{r+1} < \delta.$$

By Δ ineq,

$$\left| \mathbb{E}_n[h(G_n, o_n)] - \mathbb{E}[h(G,o)] \right|$$
$$\leq \left| \mathbb{E}_n[h(B_r^{(G_n)}(o_n))] - \mathbb{E}[h(B_r^{(G)}(o))] \right| + \varepsilon/2.$$

Q: $\exists$ only finitely many graphs of depth r then easy to approximate $h(B_r^{(G_n)}(o_n)) \approx h(B_r^{(G)}(o))$ by our assumption.

One obstruction to have finitely many graphs of depth r is that vertices can have arbitrarily large degrees. But this prob. can be made small.

Define $E_{r,k}(G,o) = \{\exists\ v \in B_r(o) \ni d_v(G) > k\}$

$E_{r,k}(G,o) \downarrow \{\exists\ v \in B_r(o) \ni d_v(G) = \infty\}$ as $k \uparrow \infty$.

$\Rightarrow \forall \eta > 0, \exists\ k \ni \mathbb{P}(E_{r,k}(G,o)) \leq \eta \quad (\mathbb{P}(\cdot) = 0)$

Consider $E_{r,k}(G,o)^c$ — $\exists$ fin. many rooted graphs $H_1, \dots, H_m \ni E_{r,k}(G,o)^c = \bigcup_{i=1}^{m} \{B_r(o) \cong H_i\}$

|||ly $E_{r,k}(G_n,o_n)^c = \bigcup_{i=1}^{m} \{B_r(o_n) \cong H_i\}$

By our assumption, we've that

$$\mathbb{P}_n(B_r(o_n) \cong H_i) \longrightarrow \mathbb{P}(B_r(o) \cong H_i) \quad 1 \leq i \leq m$$

$$\Rightarrow \mathbb{P}_n(E_{r,k}(G_n,o_n)^c) \longrightarrow \mathbb{P}(E_{r,k}(G,o)^c) \quad (*)$$

$\Rightarrow \exists\ n_0 \ni \forall\ n \geq n_0$

$$\mathbb{P}(E_{r,k}(G_n,o_n)) \leq 2\eta$$

$$\Rightarrow \left|\mathbb{E}_n[h(B_r(o_n))] - \mathbb{E}[h(B_r(o))]\right|$$

$$\leq \left|\mathbb{E}_n[h(B_r(o_n))\mathbb{1}_{E_{r,k}(G_n,o_n)^c}] - \mathbb{E}[h(B_r(o))\mathbb{1}_{E_{r,k}(G,o)^c}]\right| + 2\eta\|h\|_\infty, \quad \forall\ n \geq n_0.$$

Now since $E_{r,k}(G_n, o_n)^c$ is determined by finitely many graphs, from our assumption & (*) we obtain that $\mathbb{E}[h(B_r(o_n)) \mathbb{1}_{E_{r,k}(G_n,o_n)^c}]$

$$\longrightarrow \mathbb{E}[h(B_r(o)) \mathbb{1}_{E_{r,k}(G,o)^c}].$$

$\Rightarrow \exists\ n_1 \geq n_0 \ni \forall\ n \geq n_1$

$$\left| \mathbb{E}[h(B_r(o_n)) \mathbb{1}_{E_{r,k}(G_n,o_n)^c}] - \mathbb{E}[h(B_r(o)) \mathbb{1}_{E_{r,k}(G,o)^c}] \right| \leq \varepsilon/4$$

Choosing η small & plugging in the above bounds, the proof is complete. ∎

What about random graphs G_n?

$$\mathbb{E}_n[h(G_n, o_n)] = \frac{1}{n} \sum_{i \in [n]} h(G_n, i) \quad \text{— random variable.}$$

So how do we interpret convergence?

<u>Defn</u>: $G_n = ([n], E(G_n))$ be a seq. of RGs (possibly disconn.)

(1) $G_n \xrightarrow{LW-d} (G, o)$ if $\mathbb{E}[h(G_n, o_n)] \to \mathbb{E}_{\mu}[h(G, o)]$ $\forall$ bdd cts $h: \mathcal{G}_* \to \mathbb{R}$.

$$\mathbb{E}[h(G_n, o_n)] := \mathbb{E}\left[\mathbb{E}_n[h(G_n, o_n)]\right]$$

Exp. over G_n & o_n.

$$= \frac{1}{n}\mathbb{E}\left[\sum_{i\in[n]} h(G_n, i)\right]$$

(2) $G_n \xrightarrow{LW-p} (G, o)$ if $\mathbb{E}_n[h(G_n, o_n)] \xrightarrow{p} \mathbb{E}_\mu[h(G, o)]$ $\forall$ bdd cts h.

Rem: (1) One can define a.s. convergence as well but not so much of interest to us now. In general, not common.

(2) In LW-p, we have assumed that the limit (G, o) has prob. distribn μ on $\mathcal{G}_*$.

But we can also assume μ is a random prob measure on $\mathcal{G}_*$ i.e., μ can vary with realizations of G_n.

(Ex (A): Construct examples).

(3) Our focus will purely be with deterministic limits.

THM: Let $(G_n)_{n\geq 1}$ be a sequence of graphs (random or deterministic)

(a) $G_n \xrightarrow{LW\text{-}d} (G, o)$ iff $\forall H_* \in \mathcal{G}_*$,

$$\mathbb{E}\left[p^{(n)}(H_*)\right] = \frac{1}{n} \sum_{u \in [n]} \mathbb{P}\left(B_r^{(G_n)}(u) \cong H_*\right)$$

$$\longrightarrow \mathbb{P}\left(B_r^{(G,o)} \cong H_*\right) \quad - \quad (1)$$

$$\left[\, p^{(n)}(H_*) = \frac{1}{n} \sum_{u \in [n]} \mathbb{1}\left[B_r^{(G_n)}(u) \cong H_*\right] \right]$$

(b) $G_n \xrightarrow{LW\text{-}p} (G, o)$ iff $\forall H_* \in \mathcal{G}_*$

$$p^{(n)}(H_*) \xrightarrow{P} \mathbb{P}\left(B_r^{(G,o)} \cong H_*\right) \quad (2)$$

Rmks: (1) If G_n's are deterministic, then $p^{(n)}(H_*) = \mathbb{E}\left[p^{(n)}(H_*)\right]$

$$\Rightarrow G_n \xrightarrow{LW\text{-}d} (G, o) \Leftrightarrow G_n \xrightarrow{LW\text{-}p} (G, o).$$

(2) criteria holds for a.s. convergence also.

Proof: Follows from deterministic criteria Thm.
(Ex - complete the details). ∎

THM: Let $\mathcal{T}_* \subseteq \mathcal{G}_*$ be a subset of rooted graphs. Let $\mathcal{T}_*(r)$ be rooted graphs in $\mathcal{T}_*$ with depth/height at most r.

Assume (G,o) random graph $\ni \mathbb{P}((G,o) \in \mathcal{T}_*) = 1$.

Let $(G_n)_{n\geq 1}$ be a sequence of graphs (random/det.)

Then $G_n \xrightarrow{LW-d} (G,o)$ if (C1) holds $\forall H_* \in \mathcal{T}_*(r)$ & $r \geq 1$.

Also $G_n \xrightarrow{LWP} (G,o)$ if (C2) holds $\forall r \geq 1$ & $\forall H_* \in \mathcal{T}_*(r)$.

Proof: T.S.T. $\forall H_* \notin \mathcal{T}_*(r)$, $r \geq 1$

$$\mathbb{P}(B_r^{(G_n)}(o_n) \simeq H_*) \longrightarrow 0 = \mathbb{P}(B_r^{(G)}(o) \simeq H_*)$$

$\mathcal{G}_*(r)$ is countable.

$\mathcal{T}_*(r)$ is countable, $\forall\, \varepsilon > 0 \;\exists\, m$

& $\mathcal{T}_*(r,m) \subseteq \mathcal{T}_*(r) \ni |\mathcal{T}_*(r,m)| \le m$

$$\ni \quad \mathbb{P}\left(B_r^{(G)}(o) \in \mathcal{T}_*(r,m) \right) \ge 1 - \varepsilon.$$

$$\overline{\lim_{n\to\infty}}\, \mathbb{P}\left(B_r^{(G_n)}(o_n) \notin \mathcal{T}_*(r) \right)$$

$$= 1 - \underline{\lim_{n\to\infty}}\, \mathbb{P}\left(B_r^{(G_n)}(o_n) \in \mathcal{T}_*(r) \right)$$

$$\le 1 - \underline{\lim_{n\to\infty}}\, \mathbb{P}\left(B_r^{(G_n)}(o_n) \in \mathcal{T}_*(r,m) \right)$$

$$= 1 - \quad \mathbb{P}\left(B_r^{(G)}(o) \in \mathcal{T}_*(r,m) \right)$$

[(since $\mathcal{T}_*(r,m)$ has only finitely many graphs)
& $\mathbb{P}(B_r(o_n) \cong H_*) \longrightarrow \mathbb{P}(B_r(o) \cong H_*)$
$\forall\, H_* \in \mathcal{T}_*(r)$.]

$$\le \varepsilon$$

$$\Rightarrow \overline{\lim_{n\to\infty}}\, \mathbb{P}\left(B_r^{(G_n)}(o_n) \notin \mathcal{T}_*(r) \right) = 0$$

$\blacksquare$

Ex: Extend the proof for LW-p.

THM: Let $G_n = G(n,p)$, $p=\frac{\lambda}{n}$, $\lambda<\infty$ be the ER random graph. Let $\mathcal{T}_\lambda$ be BGW(Poisson(λ)) offspring distribution. Then

$$G_n \xrightarrow{LW-p} \mathcal{T}_\lambda.$$

Proof: $\mathbb{E}[p^{(n)}(H_*)] = \frac{1}{n}\sum_{u\in[n]} \mathbb{P}(B_r^{(G_n)}(u) \cong H_*)$

($B_r^{(G_n)}(u)$ are identically distributed in $\mathcal{G}_*$)

$$= \mathbb{P}(B_r^{(G_n)}(1) \cong H_*)$$

(shown in prev. classes) $\to \mathbb{P}(B_r^{(\mathcal{T}_\lambda)}(\emptyset) \cong H_*)$ if H_* is a tree.

Choosing $\mathcal{T}_*$ as rooted trees, we can apply the previous theorem to conclude that $G_n \xrightarrow{LW\text{-}d} \mathcal{T}_\lambda$.

Let $T \in \mathcal{T}_*(r)$, $r \geq 1$.

$$N_n(T) = n\, p^{(n)}(T) = \sum_{i \in [n]} \mathbb{1}[B_r^{(G_n)}(i) \cong T]$$

$$\frac{\mathbb{E}[N_n(T)]}{n} \longrightarrow \mathbb{P}(B_r^{(\mathcal{T}_\lambda)}(\phi) \cong T) \quad (*)$$

we want to show

$$\frac{N_n(T)}{n} \xrightarrow{P} \mathbb{P}(B_r^{(\mathcal{T}_\lambda)}(\phi) \cong T) \quad -(1)$$

STPT

$$\frac{\mathrm{Var}(N_n(T))}{\mathbb{E}[N_n(T)]^2} \longrightarrow 0. \quad -(2)$$

$$\mathbb{E}[N_n(T)^2] = n\,\mathbb{P}(B_r^{(G_n)}(1) \cong T) + n(n-1)\,\mathbb{P}(B_r^{(G_n)}(1) \cong T,\ B_r^{(G_n)}(2) \cong T)$$

[by expanding $N_n(T)^2$ & using that $(B_r^{(G_n)}(1), B_r^{(G_n)}(2)) \overset{d}{=} (B_r^{(G_n)}(i), B_r^{(G_n)}(j))$ $i \neq j$.

We'll show

$$\mathbb{P}(B_r^{(G_n)}(1) \cong T,\ B_r^{(G_n)}(2) \cong T) \to \mathbb{P}(B_r^{(T_\lambda)}(\phi) \cong T)^2 \quad (3)$$

From (3)

$$\frac{\mathrm{Var}(N_n(T))}{n^2} = \frac{\mathbb{P}(B_r^{(G_n)}(1) \cong T)}{n} + \mathbb{P}(B_r^{(G_n)}(1) \cong T,\ B_r^{(G_n)}(2) \cong T) - \mathbb{P}(\ \downarrow\ )\,\mathbb{P}(\ \downarrow\ ).$$

$\to 0 \Rightarrow (2)$

③ for $r=1$ cases in $\boxed{A1}$.

$$\mathbb{P}(B_r^n(1) \cong T,\ B_r^n(2) \cong T)$$

T1

$$= \mathbb{P}(B_r^n(1) \cong T,\ B_r^n(2) \cong T,\ d_{G_n}(1,2) > 2r)$$

$$+ \mathbb{P}(\qquad\qquad ,\ d_{G_n}(1,2) \leq 2r)$$

T2

$$\mathbb{P}(\cdots \quad ,\ d_{G_n}(1,2) \leq 2r)$$

$$\leq \mathbb{P}(B_r^{(n)}(1) \cong T,\ d_{G_n}(1,2) \leq 2r)$$

$$= \mathbb{E}\left[\mathbb{1}[B_r^n(1) \cong T]\ \mathbb{1}[2 \in B_{2r}^n(1)]\right]$$

$$= \frac{1}{n-1} \mathbb{E}\left[\mathbb{1}_{[B_r^n(1) \cong T]} \sum_{j=2}^{n} \mathbb{1}[j \in B_{2r}^n(1)]\right]$$

(since $\mathbb{1}\{j \in B_{2r}^n(1)\}$ is id. distributed)

$$\leq \frac{1}{n} \mathbb{E}\left[|B_{2r}^{(n)}(1)|\right]$$

(From A2) $\mathbb{E}\left[\left|B_{k}^{(n)}(1) \setminus B_{k-1}^{(n)}(1)\right|\right] \le \lambda^k$.

$$\frac{1}{n}\mathbb{E}\left[\left|B_{2r}^{(n)}(1)\right|\right] \le \frac{1}{n}\sum_{k=0}^{2r} \lambda^k \longrightarrow 0.$$

$\Rightarrow$ (T2) $\longrightarrow 0$. (4)

(T1) $= \mathbb{P}\left(B_r^n(1) \cong T,\ d_{G_n}(1,2) > 2r\right)$

$\mathbb{P}\left(B_r^n(2) \cong T \mid B_r^n(1) \cong T,\ d_{G_n}(1,2) > 2r\right)$

By (4) & (*)

$$\mathbb{P}\left(B_r^n(1) \cong T,\ d_{G_n}(1,2) > 2r\right) \longrightarrow \mathbb{P}\left(B_r^{(T_\lambda)}(\phi) \cong T\right)$$

Let $t = |V(T)|$.

$B_r^{(n)}(2) \mid \{B_r^{(n)}(1) \cong T,\ d_{G_n}(1,2) > 2r\}$

$\overset{d}{=} B_r^{G'_{n-t}}(2)$ (5) where G'_{n-t} is an

ER random graph on $n-t$ vertices with $p = \frac{\lambda}{n}$.

$$\Rightarrow \mathbb{P}\left(B_r^{(G_n)}(2) \cong T \,\middle|\, \cdots\cdots \right)$$

$$= \mathbb{P}\left(B_r^{(n-t)}(2) \cong T \right) \to \mathbb{P}\left(B_r^{(T_\lambda)}(\phi) \cong T \right)$$

Shows that (T1) $\longrightarrow \mathbb{P}\left(B_r^{(T_\lambda)}(\phi) \cong T \right)^2$

& ~~so~~ we have (3) ■

Explanation for (5):

$$\mathbb{P}\left(B_r^{(G_n)}(2) \cong T \,\middle|\, B_r^{(G_n)}(1) \cong T,\ d_{G_n}(1,2) > 2r \right)$$

$$= \sum_{\substack{1 \in F \subseteq [n] \\ |F| = k}} \mathbb{P}\left(B_r^{(G_n)}(2) \cong T \,\middle|\, B_r^{(G_n)}(1) \cong T,\ V_n = F,\ d_{G_n}(1,2) > 2r \right)$$
$$\times \mathbb{P}\left(V_n = F \,\middle|\, B_r^{(G_n)}(1) \cong T,\ d_{G_n}(1,2) > 2r \right)$$

$$[V_n = V(B_r^{(G_n)}(1))] \quad \text{— (6)}$$

$G_n^F = G_n \setminus F$ — ER graph on $V_n \setminus F$.

$$= \sum_{\substack{1 \in F \subseteq [n] \\ |F| = k}} \mathbb{P}\left(B_r^{(G_n^F)}(2) \cong T \,\middle|\, \cdots\cdots \right) \mathbb{P}\left(\cdot \,\middle|\, \cdots \right)$$

$$= \sum_{\substack{1 \in F \subseteq [n] \\ |F| = k}} \mathbb{P}(B_r^{(G_n^F)}(2) \cong T)\, \mathbb{P}(\cdots \mid \cdots)$$

$$= \sum_{\cdots} \mathbb{P}(B_r^{(G_{n-t})}(1) \cong T)\, \mathbb{P}(\cdots \mid \cdots)$$

$$(G_n^F \overset{d}{=} G_{n-t})$$

$$= \mathbb{P}(B_r^{(G_{n-t})}(1) \cong T)$$

$$\longrightarrow \mathbb{P}(B_r^{(\mathcal{L}_\lambda)}(\phi) \cong T) . \quad \blacksquare$$

Sparse Random Graphs : Assignment 2

Yogeshwaran D.

October 1, 2020

Submit solutions via Moodle by 11th October 10:00 PM.

1. Let G be a Cayley graph with a finite symmetric generating set S such that $o \notin S$ where o is the identity element. Let $B_n = \{v : d(v, o) \leq n\}$. Suppose that $|B_n \setminus B_{n-1}|/|B_n| \to 0$ as $n \to \infty$. Show that $B_n^{(G)}(o) \overset{LW}{\to} (G, o)$.

2. Let $G_n = G(n, p_n)$ with $p_n = \lambda/n$. Show that for all $k \geq 1$ and $n \geq \lambda k$, we have that
$$\mathbb{P}(|C(v)| > k) \leq e^{-c_\lambda k},$$
with $c_\lambda > 0$ if $\lambda < 1$.

3. Let $G_n = G(n, p_n)$ with $p_n = 1/n$. Assume that there exists $c < \infty$ such that $\mathbb{E}[|C_n(1)|] \leq cn^{1/3}$ for all n large enough. Show that for all n large enough and $a > 0$,
$$\mathbb{P}(|C_{max}| \geq an^{2/3}) \leq ca^{-2},$$
where $|C_{max}|$ is the size of the largest component in G_n.

4. Let G_n be the tree of depth k in which every vertex except the $3 \times 2^{k-1}$ leaves have degree 3 and where $n = 3(2^k - 1)$. What is the local weak limit of G_n ?

5. Let G_n be a sequence of random graphs on $[n]$ such that $G_n \overset{LW-d}{\to} (G, o)$. Let $Z_{\geq k}$ be the number of vertices i such that $|C_n(i)| \geq k$ where $C_n(i)$ is the component of i in G_n. Show that for all $k \geq 1$,
$$n^{-1}\mathbb{E}[Z_{\geq k}] \to \mathbb{P}(|C_G(o)| \geq k),$$
where $C_G(o)$ is the component of the root in (G, o).

6. Let G_n be a sequence of random graphs on $[n]$ such that $G_n \overset{LW-d}{\to} (G, o)$. Let $|C_{max}|$ be the size of the largest component in G_n. Assume that $\mathbb{P}(|C_G(o)| = \infty) = 0$, where $C_G(o)$ is the component of the root in (G, o). Show that
$$n^{-1}\mathbb{E}[|C_{max}|] \to 0.$$

ADDITIONAL PROBLEMS (not to be submitted)

1. Let $G_n = G(n, p_n)$ with $p_n = \lambda/n$. Let $B_k = \{i \in [n] : d_{G_n}(i, 1) \leq k\}$, $k \geq 1$. Show that for all $k \geq 1$,

$$\mathbb{E}[|B_k \setminus B_{k-1}|] \leq \lambda^k.$$

   Thus conclude that $\mathbb{P}(rad_{G_n}(1) \geq k) \leq e^{-c_\lambda k}$ for $\lambda < 1$ where $rad_G(1) = \max_{v \in C_G(1)} d_G(v, 1)$.

2. Construct the simplest (in your opinion) possible example where the local weak limit of a sequence of deterministic graphs is random.

13|10|20 L7 : Summary & the Giant

Summary –

- ER random graphs - sparse | dense regimes
- Thresholds for appearance of subgraphs (A 1.3)
- Our focus on sparse regime $p = \frac{\lambda}{n}$.

  # of trees $= \Theta(n)$.

  # of tree components $= \Theta(n)$ (A 1.4)

  # of k-cycles $\xrightarrow{d}$ Poisson $\left(\frac{\lambda^k}{2k}\right)$ (A 1.5)

  # of subgraphs with more than 1 cycle $= O(n^{-2})$.

  $(\deg(1), \ldots, \deg(k)) \xrightarrow{d} (Z_1, \ldots, Z_k)$

  Z_i i.i.d. Poi(λ).
- "Big insight"

  Local structure of ER random graph $\longrightarrow$ Local structure of BGW(Poi(λ)).

  $\Rightarrow$ Gives some understanding of large/giant components

Giant component — Component of size $\Omega(n)$ i.e., cn

- Formalize the above convergence via LWC or B-S convergence on space of rooted graphs.
- 2 notions of convergence of RG

$\rightarrow$ $ER(n, \frac{\lambda}{n}) \xrightarrow{LW-R} BGW(Poi(\lambda)) = \mathcal{T}_\lambda$

Local functionals of $G(n, \frac{\lambda}{n})$

$\longrightarrow$ Local fnls of $\mathcal{T}_\lambda$

(Eg. See A1.1)

- What about non-local?

Something can be said

(A2.5 – If no giant in limit, no giant in G_n.
A2.6) Good estimates for $G(n,p)$ for $\lambda < 1$.

A2.2 – Some estimates for $G(n, \frac{1}{n})$.

A2.3

Can we understand giant better when $\exists$ an ∞ component in the limit?

What other non-local fnls can we study?

GIANT COMPONENT UNDER LWC

(See Section 2.5 of vdH-2 for details & all results)

We'll see some highlights.

THM: (upper bound on the giant)

Let G_n denote a finite random graph (may be disconn.) $G_n \xrightarrow{LW-P} (G, o)$. $\zeta = \mathbb{P}(|\mathcal{C}(o)| = \infty)$, $\mathcal{C}(o)$ - component of root o. Then $\forall\ \varepsilon > 0$

$$\mathbb{P}(|\mathcal{C}_{max}| \geq n(\zeta + \varepsilon)) \to 0.$$

Proof: (sketch - Extension of A2.5, A2.6).

$\mathcal{C}(v)$ - Component of v.

$\mathcal{C}_i$ - i^{th} largest component (ties broken arbitrarily)

$$Z_{\geq k} = \sum_{v \in [n]} \underbrace{\mathbb{1}\left[|\mathcal{C}(v)| \geq k\right]}_{= h(G_n, v)} \qquad \mathbb{E}_n[h(G_n, o_n)] = \frac{Z_{\geq k}}{n}$$

$$G_n \xrightarrow{LW-P} (G, o) \Rightarrow \frac{Z_{\geq k}}{n} \xrightarrow{P} \zeta_{\geq k} := \mathbb{P}(|\mathcal{C}_G(o)| \geq k)$$

$$\{|\mathcal{C}_{max}| \geq k\} = \{Z_{\geq k} \geq k\}$$

If $Z_{\geq k} \geq 1$, $\quad |\mathcal{C}_{max}| \leq Z_{\geq k}$

$$\& \quad \zeta_{\geq k} \downarrow \zeta = \mathbb{P}(|\mathcal{C}_G(o)| = \infty)$$

$$\Rightarrow \mathbb{P}(|\mathcal{C}_{max}| \geq n(\zeta + \varepsilon)) \to 0 \qquad \blacksquare$$

When does $n^{-1}|\mathcal{C}_{max}| \xrightarrow{P} \zeta$? [If $\zeta = 0$, A2.6]

THM 1. G_n random graph on $[n]$ & $G_n \xrightarrow{LWP} (G, O)$.
Let $\zeta = \mathbb{P}(|\mathcal{C}_G(O)| = \infty) > 0$.
Assume that

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n^2} \mathbb{E}\left[\#\{x, y \in [n] : |\mathcal{C}_G(x)|, |\mathcal{C}_G(y)| \geq k, x \nleftrightarrow y\}\right] = 0. \quad (*)$$

Then

$$\frac{|\mathcal{C}_{max}|}{n} \xrightarrow{P} \zeta, \quad \frac{|\mathcal{C}_2|}{n} \xrightarrow{P} 0.$$

Necessary & sufficiency of (*) will be in Assignments.

$X_{n,k} = o_{k,p}(1)$ if

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}} \mathbb{P}(|X_{n,k}| > \varepsilon) = 0$$

LEMMA Under assumptions of thm 1

$$\frac{1}{n^2} \sum_{i\geq 1} |\mathcal{C}_i|^2 \xrightarrow{P} \zeta^2$$

&

$$\frac{1}{n^2} \sum_{i\geq 1} |\mathcal{C}_i|^2 \mathbb{1}[|\mathcal{C}_i| \geq k] = \zeta^2 + o_{k,p}(1)$$

Proof: $\frac{Z_{\geq k}}{n} = \frac{1}{n} \sum_{i\geq 1} |\mathcal{C}_i| \mathbb{1}[|\mathcal{C}_i| \geq k] = \zeta + o_{k,p}(1)$

$$\frac{1}{n^2} \sum_{\substack{i,j\geq 1 \\ i\neq j}} |\mathcal{C}_i||\mathcal{C}_j| \mathbb{1}[|\mathcal{C}_i| \geq k, |\mathcal{C}_j| \geq k]$$

$$= \frac{1}{n^2} \#\{x, y \in [n] : |\mathcal{C}_G(x)|, |\mathcal{C}_G(y)| \geq k, x \nleftrightarrow y\}$$

$$\overset{\text{(Assumption \& Markov's ineq.)}}{=} o_{k,p}(1)$$

$$\frac{1}{n^2} \sum_{i\geq 1} |\mathcal{C}_i|^2 \mathbb{1}[|\mathcal{C}_i| \geq k] = \left(\frac{1}{n} \sum_{i\geq 1} |\mathcal{C}_i| \mathbb{1}[|\mathcal{C}_i| \geq k]\right)^2 + o_{k,p}(1)$$

$$= \tau_1^2 + o_{k,p}(1).$$

$$\frac{1}{n^2}\sum_{i\geq 1}|\mathcal{C}_i|^2 = \frac{1}{n^2}\sum_{i\geq 1}|\mathcal{C}_i|^2 \mathbb{1}[|\mathcal{C}_i|\geq k] + O\left(\frac{k}{n}\right)$$

$$= \tau_1^2 + o_{k,p}(1).$$

$$\left[\frac{1}{n^2}\sum_{i\geq 1}|\mathcal{C}_i|^2 \mathbb{1}[|\mathcal{C}_i|<k] \leq \frac{k}{n}\left(\frac{1}{n}\sum_{i\geq 1}|\mathcal{C}_i|\right)\right]$$

Proof of THM 1:

Idea of proof: $q_{i,n} := \dfrac{|\mathcal{C}_i|\mathbb{1}[|\mathcal{C}_i|\geq k]}{\sum_{i\geq 1}|\mathcal{C}_i|\mathbb{1}[|\mathcal{C}_i|\geq k]}$

$$\sum_{i\geq 1} q_{i,n} = 1 \qquad \sum q_{i,n}^2 = 1 + o_{k,p}(1)$$

(prev. lemma)

$$\Rightarrow \max_{i\geq 1} q_{i,n} = 1 + o_{k,p}(1)$$

$$\Rightarrow q_{1,n} = 1 + o_{k,p}(1), \quad q_{2,n} = o_{k,p}(1)$$

$$\Rightarrow \frac{|\mathcal{C}_{max}|}{n} = \tau_1 + o_{k,p}(1), \quad \frac{|\mathcal{C}_2|}{n} = o_{k,p}(1)$$

$x_i = |\mathcal{C}_i|\mathbb{1}[|\mathcal{C}_i|\geq k]/n$

By prev. lemma & LWC-p, w.h.p.

$$\sum_{i\geq 1} x_i \leq \tau_1 + \varepsilon/4 \qquad -① $$

$$\sum_{i\geq 1} x_i^2 \geq \tau_1^2 - \varepsilon^2 \qquad -②$$

[A_n w.h.p., $\mathbb{P}(A_n) \to 1$]

If $x_{1,n} \leq \zeta_1 - \varepsilon$ for some $\varepsilon > 0$.

i.e., $\overline{\lim}\, \mathbb{P}(x_{1,n} \leq \zeta_1 - \varepsilon) > 0$

$$\wedge|$$

by ① $\overline{\lim}\, \mathbb{P}\left(\sum_{i\geq 1} x_{i,n}^2 \leq (\zeta_1 - \varepsilon)^2 + \left(\frac{5\varepsilon}{4}\right)^2\right)$

$$\wedge|$$

$$\overline{\lim}\, \mathbb{P}\left(\sum_{i\geq 1} x_{i,n}^2 \leq \zeta_1^2(1-\varepsilon)\right)$$

If ε is small, this contradicts ②

so $\overline{\lim}\, \mathbb{P}(x_{1,n} \leq \zeta_1 - \varepsilon) = 0$.

$$\Rightarrow \lim \mathbb{P}(x_{1,n} \geq \zeta_1 - \varepsilon) = 1.$$

We've shown that

$$\lim \mathbb{P}(x_{1,n} \leq \zeta_1 + \varepsilon) = 1.$$

$$\Rightarrow \frac{|\mathcal{C}_{max}|}{n} \xrightarrow{\mathbb{P}} \zeta_1.$$

$$\sum_{i\geq 1} x_i^2 = \zeta_1 + o_{\mathbb{P}}(1)$$

$$\& \quad x_1 = \zeta_1 + o_{\mathbb{P}}(1)$$

$\Rightarrow x_2 = o_{\mathbb{P}}(1)$. [Fill the gap!] ∎

A sequence $(X_n)_{n\geq 1}$ of r.v. is called uniformly integrable (UI) if

$$\lim_{K\to\infty} \overline{\lim_{n\to\infty}}\, \mathbb{E}\left[|X_n| \mathbb{1}\{|X_n| > K\}\right] = 0.$$

If $\sup_{n\geq 1} \mathbb{E}[|X_n|^2] < \infty$

$$\mathbb{E}[|X_n| \mathbb{1}[|X_n|>K]] \overset{CS}{\leq} \sqrt{\mathbb{E}[X_n^2]} \sqrt{\mathbb{P}(|X_n|>K)}$$

$$\leq \frac{1}{\cancel{K}} \mathbb{E}[X_n^2]$$

THM 2: Same assumptions as in THM 1.

$v_\ell(\mathcal{C}_{max})$ = # of vertices of degree ℓ in $\mathcal{C}_{max}$

$$\frac{v_\ell(\mathcal{C}_{max})}{n} \xrightarrow{P} \mathbb{P}(|\mathcal{C}(0)| = \infty, d_0 = \ell)$$

d_0 = deg of 0 in $(G, 0)$

If $\{d_{o_n}^{(n)}\}_{n\geq 1}$ is U.I. then

# of edges in $\mathcal{C}_{max}$ — $$\frac{E(\mathcal{C}_{max})}{n} \to \frac{1}{2} \mathbb{E}[\mathbb{1}[|\mathcal{C}(0)| = \infty] \times d_0]$$

THM 3: Let $G_n = G(n,p)$, $p = \frac{\lambda}{n}$.

Then

$$\frac{|\mathcal{C}_{max}|}{n} \xrightarrow{P} \zeta_\lambda, \quad \frac{|\mathcal{C}_2|}{n} \xrightarrow{P} 0$$

where $\zeta_\lambda = 1 - p_{ext}(\lambda)$ for BGW(Poi(λ))

$$\frac{v_\ell(\mathcal{C}_{max})}{n} \xrightarrow{P} e^{-\lambda} \frac{\lambda^\ell}{\ell!} [1 - p_{ext}(\lambda)^\ell]$$

$$\frac{E(\mathcal{C}_{max})}{n} \xrightarrow{P} \frac{1}{2} \lambda [1 - p_{ext}(\lambda)^2]$$

Proof sketch: We'll verify Assumption in THM 1 for Erdos-Renyi RG.

Since $G_n \xrightarrow{d} (\mathcal{T}_\lambda, \phi)$ & so from thm 2, we've to compute

$$\mathbb{P}(|\mathcal{T}_\lambda| = \infty, d_\phi = l) = ?$$

$$\mathbb{E}[d_0 \mathbb{1}[|\mathcal{T}_\lambda| = \infty]] = ?$$

If $d_0 = l$ & $|\mathcal{T}_\lambda| = \infty$ then one of the 'l' subtrees at child of ϕ is infinite. Subtree survival is indep of d_0.

Complete the proof...... ☒

$\Rightarrow$ For ER graph, $\exists!$ giant component for $\lambda > 1$ & else no giant component.

2017 : L 8 - Structure of Giant

THM 2: Same assumptions as in THM 1.

$v_\ell(\mathcal{C}_{max}) = \#$ of vertices of degree ℓ in $\mathcal{C}_{max}$

$$\frac{v_\ell(\mathcal{C}_{max})}{n} \xrightarrow{P} \mathbb{P}(|\mathcal{C}(0)| = \infty, d_0 = \ell)$$

$d_0 = $ deg of 0 in $(G, 0)$

If $\{d_{o_n}^{(n)}\}_{n \geq 1}$ is U.I. then

$$\frac{\overbrace{E(\mathcal{C}_{max})}^{\text{\# of edges in } \mathcal{C}_{max}}}{n} \longrightarrow \frac{1}{2} E\left[\mathbb{1}[|\mathcal{C}(0)| = \infty] \times d_0\right]$$

Proof: $A \subseteq \mathbb{N}$. d_v - $\deg(v)$ in G_n

$$Z_{A, \geq k} = \sum_{v \in [n]} \mathbb{1}[|\mathcal{C}(v)| \geq k, d_v \in A]$$

$$G_n \xrightarrow{LW\text{-}P} (G, 0) \Rightarrow \frac{Z_{A, \geq k}}{n} \xrightarrow{P} \mathbb{P}(|\mathcal{C}(0)| \geq k, d_0 \in A) \to (1)$$

Since $\mathbb{P}(|\mathcal{C}_{max}| \geq k) \to 1 \quad \forall\, k \geq 1$

('cos $n^{-1} |\mathcal{C}_{max}| \xrightarrow{P} \zeta > 0$)

$$\mathbb{P}\left(\frac{1}{n} \sum_{a \in A} v_a(\mathcal{C}_{max}) \leq \frac{Z_{A, \geq k}}{n}\right) \to 1. \quad -(2)$$

$A = \{\ell\}^c$. $v_a(\mathcal{C}_{max}) = \#\{v \in \mathcal{C}_{max} : d_v = a\}$

$$\mathbb{P}\left(\frac{1}{n}\left[|\mathcal{C}_{max}| - v_\ell(\mathcal{C}_{max})\right] \leq \mathbb{P}(|\mathcal{C}(0)| \geq k, d_0 \neq \ell) + \varepsilon/2\right) \to 1. \quad -(3)$$

From (1) & (2)

$$\Rightarrow \mathbb{P}\left(\frac{v_\ell(\mathcal{C}_{max})}{n} \geq \mathbb{P}(|\mathcal{C}_\varphi(o)|=\infty) - \mathbb{P}(|\mathcal{C}(o)| \geq k, d_o \neq \ell) - \varepsilon/2 \right) \to 1.$$

$$\frac{|\mathcal{C}_{max}|}{n} = \frac{1}{n}\left[|\mathcal{C}_{max}| - v_\ell(\mathcal{C}_{max})\right] + v_\ell(\mathcal{C}_{max})$$

(using ③) $\leq \mathbb{P}(|\mathcal{C}(o)| \geq k, d_o \neq \ell) + v_\ell(\mathcal{C}_{max}) + \varepsilon/2$

The above event happens w.p. $\to 1$.

Also choose k large $\ni$ $\mathbb{P}(|\mathcal{C}(o)| \geq k, d_o \neq \ell) - \mathbb{P}(|\mathcal{C}(o)| = \infty, d_o \neq \ell) \leq \varepsilon$.

Further, Thm 1 $\Rightarrow$ $\mathbb{P}\left(\frac{|\mathcal{C}_{max}|}{n} \geq \zeta_\varphi - \varepsilon\right) \to 1$

where $\zeta_\varphi = \mathbb{P}(|\mathcal{C}_\varphi(o)| = \infty) > 0$ (by assumption)

$$\Rightarrow \mathbb{P}\left(\frac{v_\ell(\mathcal{C}_{max})}{n} \geq \zeta_\varphi - \mathbb{P}(|\mathcal{C}_\varphi(o)| = \infty, d_o \neq \ell) - \varepsilon/2\right) \to 1$$

$$\Rightarrow \mathbb{P}\left(\frac{v_\ell(\mathcal{C}_{max})}{n} \geq \mathbb{P}(|\mathcal{C}_\varphi(o)| = \infty, d_o = \ell) - \varepsilon/2\right) \to 1$$

Also use ② for $A = \{\ell\}$ & noting that it holds $\forall k$,

$$\mathbb{P}\left(\frac{v_\ell(\mathcal{C}_{max})}{n} \leq \mathbb{P}(|\mathcal{C}_\varphi(o)| = \infty, d_o = \ell) + \varepsilon\right) \to 1$$

$$\Rightarrow \frac{v_\ell(\mathcal{C}_{max})}{n} \xrightarrow{\mathbb{P}} \mathbb{P}(|\mathcal{C}(o)| = \infty, d_o = \ell).$$

$$|E(\mathcal{C}_{max})| = \frac{1}{2}\sum_{\ell \geq 1} \ell\, v_\ell(\mathcal{C}_{max}) = \frac{1}{2}\sum_{v \in [n]} d_v$$

$$\frac{|E(\mathscr{C}_{max})|}{n} = \frac{1}{2n}\sum_{l \le k} l\, v_l(\mathscr{C}_{max}) \quad \text{— (T1)}$$

$$+ \frac{1}{2n}\sum_{l > k} l\, v_l(\mathscr{C}_{max}) \quad \text{— (T2)}$$

(T1) $\xrightarrow{P} \frac{1}{2}\sum_{l \le k} \mathbb{P}(|\mathscr{C}(o)| = \infty, d_o = l)\, l$

$$= \frac{1}{2}\mathbb{E}\left[d_o \mathbb{1}[|\mathscr{C}(o)| = \infty, d_o \le k]\right]$$

$n_l = v_l(G_n) = \#\{v \in [n] : d_v = l\} \ge v_l(\mathscr{C}_{max})$

$\Rightarrow$ (T2) $\le \frac{1}{2}\sum_{l > k}\frac{l\, n_l}{n} = \mathbb{E}_n\left[\frac{d_{o_n}^{(n)}}{2}\mathbb{1}[d_{o_n}^{(n)} > k]\right]$

o_n – random root in G_n

By U.I., $\forall\, \varepsilon > 0$,

$$\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\, \mathbb{P}\left(\mathbb{E}_n\left[d_{o_n}^{(n)}\mathbb{1}[d_{o_n}^{(n)} > k]\right] > \varepsilon\right).$$

(Markov's ineq.) $\le \lim_{k\to\infty}\overline{\lim_{n\to\infty}} \dfrac{\mathbb{E}\left[\mathbb{E}_n\left[d_{o_n}^{(n)}\mathbb{1}[d_{o_n}^{(n)} > k]\right]\right]}{\varepsilon}$

$\le \lim_{k\to\infty}\overline{\lim_{n\to\infty}}\, \varepsilon^{-1}\mathbb{E}\left[d_{o_n}^{(n)}\mathbb{1}[d_{o_n}^{(n)} > k]\right] = 0$ (by U.I.)

Complete proof by letting $k \uparrow \infty$ in (T1). ∎

Remains to prove for $G(n,p)$ —

$$\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\frac{1}{n^2}\mathbb{E}\left[\#\{x, y \in [n] : |\mathscr{C}(x)|, |\mathscr{C}(y)| \ge k,\ x \not\leftrightarrow y\}\right] = 0. \quad \text{— (*)}$$

LEMMA: Assume other conditions than ⊛ in THM 1.
Then ⊛ holds if

(i) $\lim_{r\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n^2} \mathbb{E}\left[\#\{x,y \in [n] : |\partial B_r(x)|, |\partial B_r(y)| \geq r, x \not\leftrightarrow y\}\right] = 0$

(ii) $\exists\ r = r_k \to \infty \ni (G,0)$ satisfies the following -

$$\mathbb{P}(|\mathcal{C}_G(0)| \geq k, |\partial B_r(0)| < r_k) \to 0$$
$$\mathbb{P}(|\mathcal{C}(0)| < k, |\partial B_r(0)| \geq r_k) \to 0.$$

Rmk: if $r_k = k$, $\mathbb{P}(|\mathcal{C}_G(0)| < k, |\partial B_r(0)| \geq r) = 0.$

For GW tree with mean $\lambda > 1$,

Ex. (A) $\lambda^{-r} \mathbb{E}\left[|\partial B_r(\phi)| \mid |\mathcal{C}_G(\phi)| = \infty\right] \xrightarrow{a.s.} M$ for $M > 0$.

Conditioned on survival, $|\partial B_r(\phi)|$ grows exp. fast!
$\Rightarrow$ first part of (ii) holds.

Note $\lambda^{-r} \mathbb{E}\left[|\partial B_r(\phi)|\right] = 1.$

Proof: $P_r^{(2)} = \#\{x,y : |\partial B_r(x)|, |\partial B_r(y)| \geq r, x \not\leftrightarrow y\}$

$\theta_k^{(2)} = \#\{x,y : |\mathcal{C}_G(x)|, |\mathcal{C}_G(y)| \geq k, x \not\leftrightarrow y\}.$

$$|P_r^{(2)} - \theta_k^{(2)}| \leq 2n(Z_1 + Z_2)$$

$$Z_1 = \#\{v : |\partial B_r(v)| < r, |\mathcal{C}_G(v)| \geq k\}$$

$$Z_2 = \#\{v : |\partial B_r(v)| \geq r, |\mathcal{C}_G(v)| < k\}$$

$$G_n \xrightarrow{LW\text{-}P} (G, o) \implies$$

$$\frac{Z_1 + Z_2}{n} \xrightarrow{P} \mathbb{P}(|\mathcal{C}_o(o)| \geq k, |\partial B_r(o)| < r)$$
$$+ \mathbb{P}(|\mathcal{C}(o)| < k, |\partial B_r(o)| \geq r) \quad -(1)$$

Note that
$$\frac{1}{n^2} |P_r^{(2)} - Q_k^{(2)}| \leq 2 \frac{(Z_1 + Z_2)}{n}$$

$$\frac{Z_1}{n} \leq 1, \quad \frac{Z_2}{n} \leq 1 \quad \& \quad \frac{Z_1 + Z_2}{n} \xrightarrow{P} \ldots$$

By DCT for convergence in prob./measure,

we have that $\overline{\lim_{n \to \infty}} \frac{\mathbb{E}[|P_{r_k}^{(2)} - Q_k^{(2)}|]}{n^2}$

$$\leq 2 [\text{RHS of } (1)]$$

Now by Assumption (ii),

$$\lim_{k \to \infty} \overline{\lim_{n \to \infty}} \frac{\mathbb{E}[|P_{r_k}^{(2)} - Q_k^{(2)}|]}{n^2} = 0$$

$$\lim_{k \to \infty} \overline{\lim_{n \to \infty}} \frac{\mathbb{E}[Q_k^{(2)}]}{n^2} \leq \lim_{k \to \infty} \overline{\lim_{n \to \infty}} \frac{\mathbb{E}[P_{r_k}^{(2)}]}{n^2}$$

$$= 0 \quad \text{(by Assumption (i))}$$

$\blacksquare$

Let O_1, O_2 be 2 vertices chosen uniformly at random & indep. in $[n]$.

$$\frac{1}{n^2}\mathbb{E}\left[\#\{x,y : |\partial B_r(x)|, |\partial B_r(y)| \geq r, \ x \not\leftrightarrow y\}\right]$$

$$= \mathbb{P}\left(|\partial B_r(O_1)|, |\partial B_r(O_2)| \geq r, \ O_1 \not\leftrightarrow O_2\right)$$

$$= \sum_{\substack{b_0^1, b_0^2 \geq r \\ s_0^1, s_0^2}} \mathbb{P}\left(O_1 \not\leftrightarrow O_2, |\partial B_r(O_i)| = b_0^i, |B_{r-1}(O_i)| = s_0^i, i = 1,2\right)$$

$\partial B_r(O_i)$ — vertices having an edge to a vertex in $B_{r-1}(O_i)$ but not connected to O_2

$$|\partial B_r(O_1)| \leq \mathrm{Bin}\big(\underbrace{n - s_0^{(1)} - s_0^{(2)}}_{\substack{\text{choice of vertices} \\ \text{available to } B_{r-1}(O_i)}}, p\big)$$

large boundary of O_1 $\Rightarrow$ more choices for edges

$\Rightarrow$ less likely to miss a vertex in $\mathscr{C}(O_2)$ if $\partial B_r(O_2)$ is large.

See vdH Vol. 2, Sec 2.5. for more details.

$$\mathbb{P}(d_{G_n}(O_1, O_2) \le k) = \mathbb{E}\left[\mathbb{1}[O_2 \in B_k(O_1)] \right]$$

O_2 is random vertex

$$= \mathbb{E}\left[\frac{1}{n} \sum_{v \in [n]} \mathbb{1}[v \in B_k(O_1)] \right]$$

F-T theorem

$$= \frac{1}{n} \mathbb{E}\left[|B_k(O_1)| \right]$$

$$= \frac{1}{n} \mathbb{E}\left[|B_k(1)| \right]$$

$$= \frac{1}{n} \sum_{l=0}^{k} \lambda^l = \frac{\lambda^{k+1} - 1}{n(\lambda - 1)}$$

for $\lambda > 1$.

If $k = \dfrac{(1-\varepsilon)\log n}{\log \lambda}$

then $\mathbb{P}\left(d_{G_n}(O_1, O_2) \le \dfrac{(1-\varepsilon)\log n}{\log \lambda} \right) \longrightarrow 0.$

$\Rightarrow$ Two random vertices are at distance $\ge \log_\lambda n$

THM (THM 2.26 of vd H vol. 2)

Conditioned on $O_1 \longleftrightarrow O_2$,

$$\frac{d_{G_n}(O_1, O_2)}{\log n} \xrightarrow{P} \frac{1}{\log \lambda}$$

27/10: L9 - TIGHTNESS CRITERION.

What techniques are available to show

$G_n \xrightarrow{LWC} (G, o)$? Analysis of BFS.

Some general metric space techniques?

Let X_n be random elements in a m. space (S, d).

We can talk of $X_n \xrightarrow{d} X$ or $\mathbb{P}_n \longrightarrow \mathbb{P}$

where $\mathbb{P}_n(\cdot) = \mathbb{P}(X_n \in \cdot)$.

How to show convergence of a deterministic sequence say $(x_n)_{n\geq 1}$?

- S.T. for any subsequence $\{n_k\}_{k\geq 1}$ $\exists$ a further subsequence $\{n_k'\}_{k\geq 1}$ $\ni$ $x_{n_k'} \to x$ as $n \to \infty$.

Equivalently
- S.T. any subsequence has a convergent subsequence (Relative compactness)
- S.T. subsequential limits are unique (Uniqueness).

Now we develop such a criterion for $\mathbb{P}_n$'s or X_n's.

LEMMA: If for any sequence $(n_k)_{k\geq 1}$, $\exists$ a further subsequence $n_{k(r)} \ni \mathbb{P}_{n_{k(r)}} \xrightarrow{d} \mathbb{P}$ then $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$

(Here $\mathbb{P}_n$ & $\mathbb{P}$ are prob. measures.)

Proof: If not true, then for some $f \in C_b(S)$ & $\varepsilon > 0$ $\exists$ a subsequence $n_k \ni \left| \int f \, d\mathbb{P}_{n_k} - \int f \, d\mathbb{P} \right| > \varepsilon$.

a $\Rightarrow\Leftarrow$ to $\exists$ a subsequence of $n_k \ni \mathbb{P}_{n_{k(r)}} \xrightarrow{d} \mathbb{P}$.

To prove $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$, suffices to show two things.

(1) S.T. for any sequence $\mathbb{P}_{n_k}$ $\exists$ a converging subsequence $\mathbb{P}_{n_{k(r)}}$

(2) S.T. all subsequential limits are equal.

(1) — relative compactness. $\exists$ good criteria to show (1).

(2) — Often Ad-hoc.

Def: (Tightness): A family of prob measures on (S,d) $(P_i)_{i\in I}$ (I - index set, arbitrary) is said to be tight if for any $\varepsilon > 0$ $\exists$ a compact $K \subseteq S$ $\ni \sup_{i \in I} \mathbb{P}_i(K^c) < \varepsilon$.

Of course $\{\mathbb{P}\}$ is tight if $\exists$ a sequence of cpt sets $K_n \uparrow S$.

Then $\mathbb{P}(K_n) \uparrow \mathbb{P}(S) = 1 \Rightarrow \exists\, n \ni \mathbb{P}(K_n) \geq 1 - \varepsilon$

$\Updownarrow$

$\mathbb{P}(K_n^c) \leq \varepsilon$.

SELECTION THM: If $(\mathbb{P}_n)_{n\geq 1}$ is a tight sequence of prob. measures on (S,d), then $(\mathbb{P}_n)_{n\geq 1}$ is rel. compact i.e.,

$\forall$ subsequences n_k, $\exists$ a further subsequence $n_k' \ni$

$\mathbb{P}_{n_k'} \xrightarrow{d} \mathbb{P}$ for some $\mathbb{P}$. ($\mathbb{P}$ can depend on subsequence)

Cantor's Diagonalization: A countable set & $f_n : A \to \mathbb{R}$ $\forall\, n \geq 1$. Then $\exists$ a subsequence $n_k \ni f_{n_k}(a) \to f(a) \in [-\infty, \infty]$ $\forall\, a \in A$.

Proof: $(S = \mathbb{R})$. [Proof for general case, See DP THM 3.2 or wait for Stoch Process course!]

Let A be a countable dense subset of $\mathbb{R}$. Consider the CDFs F_n for $\mathbb{P}_n$.

By diagonalization, $\exists$ a subsequence $n_k \ni F_{n_k}(a) \to F(a) \ \forall a \in A$.

Here F is some fn. To complete the proof, we've to show that F is a CDF & convergence holds at all cty points of F.

$\bar{F}(x) = \inf\{F(a) : a > x, a \in A\}$

By construction, F is non-decreasing on A & so is $\bar{F}$ on $\mathbb{R}$.

$$\bar{F}(x) = \inf\{F(a) : x < a, a \in A\} = \inf_{y > x} \inf\{F(a) : a > y, a \in A\}$$
$$= \inf\{\bar{F}(y) : y > x\}$$

$\Rightarrow \bar{F}$ is right cts.

$$\text{Tightness} \Rightarrow \exists a_1, a_2 \in A \ni \mathbb{P}_n(\underbrace{(a_1, a_2]}_{= F_n(a_2) - F_n(a_1)}) \geq 1 - \varepsilon \quad \forall n \geq 1$$

$$\Rightarrow \bar{F}(a_2) - \bar{F}(a_1) \geq 1 - \varepsilon. \Rightarrow \lim_{x \to \infty} \bar{F}(x) - \lim_{x \to -\infty} \bar{F}(x) = 1$$

Since $\bar{F}$ non-decreasing, limits exist & $\bar{F} \in [0, 1]$

$$\Rightarrow \lim_{x \to \infty} \bar{F}(x) = 1 = 1 - \lim_{x \to -\infty} \bar{F}(x).$$

$\Rightarrow \bar{F}$ is a CDF.

If x is a pt of cty of $\bar{F}$ then $\lim_{\substack{a \uparrow x \\ a \in A}} F(a) = \bar{F}(x)$ & $\lim_{\substack{b \downarrow x \\ b \in A}} F(b) = \bar{F}(x)$. — (1)

Let x be a cty pt of $\bar{F}$ & choose $a < x < b$, $a, b \in A$.

$$F(a) = \lim_{k \to \infty} F_{n_k}(a) \leq \underline{\lim}_{k \to \infty} F_{n_k}(x) \leq \overline{\lim}_{k \to \infty} F_{n_k}(x)$$

(non-decreasing of F_{n_k}'s)

$$\underset{(\text{non-decreasing})}{\leq} \overline{\lim}_{k \to \infty} F_{n_k}(b) = F(b).$$

From (1), we get $\overline{\lim} F_{n_k}(x) = \underline{\lim} F_{n_k}(x) = \bar{F}(x)$.

for $S = \mathbb{R}$.

General S (proof sketch)

From Stone-Weierstrass thm, $C_b(K)$ is seperable for a cpt set K under supnorm.

$C_b(K) = C(K)$, cts fns on K.

$\mathbb{P}_n$ are tight $\Rightarrow \forall\, r \geq 1$, $\exists$ cpt $K_r \ni$

$$\mathbb{P}_n(K_r) > 1 - \frac{1}{r} \quad \forall\, n \geq 1.$$

Let $C_r \subset C(K_r)$ be a countable dense subset.

By diagonalization, $\exists$ a subsequence $n_k \ni$

$$\int f \, d\mathbb{P}_{n_k} \longrightarrow \int f \, d\mathbb{P}^r \quad \forall\, f \in C_r.$$

Since C_r is dense, $\int f \, d\mathbb{P}_{n_k} \longrightarrow \int f \, d\mathbb{P}^r \quad \forall\, f \in C(K_r)$

$$\left[\left| \int f \, d\mathbb{P} - \int g \, d\mathbb{P} \right| \leq \int |f-g| \, d\mathbb{P} \leq \|f-g\|_\infty \right]$$

$\forall\, f \in C_b(S)$

$$\left| \int f \, d\mathbb{P}_{n_k} - \int_{K_r} f \, d\mathbb{P}_{n_k} \right| \leq \int_{K_r^c} f \, d\mathbb{P}_{n_k} \leq \|f\|_\infty \mathbb{P}_{n_k}(K_r^c) \leq \frac{\|f\|_\infty}{r}$$

$\Rightarrow \exists$ a limit $I(f) := \lim_{k \to \infty} \int f \, d\mathbb{P}_{n_k}$

$$\Big[\because \left| \int f \, d\mathbb{P}_{n_k} - \int f \, d\mathbb{P}_{n_\ell} \right| \leq \left| \int f \, d\mathbb{P}_{n_k} - \int_{K_r} f \, d\mathbb{P}_{n_k} \right|$$

$$+ \left| \int_{K_r} f \, d\mathbb{P}_{n_k} - \int_{K_r} f \, d\mathbb{P}_{n_\ell} \right|$$

$$+ \left| \int_{K_r} f \, d\mathbb{P}_{n_\ell} - \int f \, d\mathbb{P}_{n_\ell} \right|$$

$$\leq \frac{2\|f\|_\infty}{r} + \left| \int_{K_r} f \, d\mathbb{P}_{n_k} - \int_{K_r} f \, d\mathbb{P}_{n_\ell} \right| \Big]$$

To complete the proof, we need to show that $\exists$ a $\mathbb{P}$ $\ni$ $I(f) = \int f \, d\mathbb{P}$. One can show that I is a pos. lin. fnl & use Riesz representation thm $I(f \mathbb{1}_{K_r}) = \int_{K_r} f \, d\mathbb{P}^r$ ∎

Prohorov's theorem.

(S,d) is a Polish space. (X_n) is rel. compact $\Rightarrow$ $(X_n)_{n\geq 1}$ is tight.

THM (THM 2.6 of vdH-2).

Let G_n be a sequence of graphs on $[n]$ & o_n the random root chosen uniformly. If $(d^{(n)}_{o_n})$ is uniformly integrable $\left(\lim_{K\to\infty} \overline{\lim_{n\to\infty}} \mathbb{E}\left[d^{(n)}_{o_n} \mathbb{1}[d^{(n)}_{o_n} \geq K]\right] = 0\right)$ then $(G_n, o_n)_{n\geq 1}$ is tight.

LEMMA (compactness:)

Let $h: \mathbb{N} \to \mathbb{N}$ be an increasing fn.

Then $\mathcal{K}_h = \{(G,o) : |B^G_r(o)| \leq h(r) \ \forall r \geq 1\}$ is compact.

<u>Proof:</u> $\forall r \geq 1$, $\exists$ finitely many equivalence classes of rooted graphs $F_{r,1}, \ldots, F_{r,n_r} \ni |B^{F_{r,i}}_r(o)| \leq h(r)$ i.e., if $F \ni$ $|B^F_r(o)| \leq h(r)$ then $B^F_r(o) \cong F_{r,i}$ for some $1 \leq i \leq n_r$.

$$\Rightarrow \mathcal{K}_h \subseteq \bigcup_{i=1}^{n_r} \left\{(G,o) : d_{\mathcal{G}_*}((G,o), F_{r,i}) \leq \frac{1}{r+1}\right\}$$

<u>Proof of thm:</u> $$\mathcal{K}_h^c = \bigcup_{r\geq 1} \{(G,o) : |B^G_r(o)| > h(r)\}$$
$$\subseteq \bigcup_{r\geq 1} \{(G,o) : \exists v \in B^G_r(o) \ni d_v > M_r\}$$
$$= \bigcup_{r\geq 1} F^M_r(o) \text{ for } M: \mathbb{N} \to \mathbb{N} \text{ increasing fn.}$$

ETPT $\forall \varepsilon > 0$ & $\forall r \geq 1$ $\exists M < \infty \ni \sup_n \mathbb{P}((G_n, o_n) \in F^M_r) \leq \varepsilon$ — (1)

$\Rightarrow$ choose $\frac{\varepsilon}{2^r}$ for $r \geq 1$; $\exists M < \infty \ni \sup_n \mathbb{P}((G_n, o_n) \in F^M_r) \leq \frac{\varepsilon}{2^r}$

$\sup_n \mathbb{P}((G_n, o_n) \in \mathcal{K}^c_h) \leq \varepsilon \Rightarrow$ Tightness of (G_n, o_n).

Let $f(d) = \sup_n \mathbb{E}\left[d^{(n)}_{o_n} \mathbb{1}[d^{(n)}_{o_n} > d]\right]$

U.I. of $d^{(n)}_{o_n}$ $\Rightarrow \lim_{d\to\infty} f(d) = 0$.

$m(G_n) = \mathbb{E}[d^{(n)}_{o_n}] \leq f(0) < \infty$; $m(G_n) = \frac{1}{n}\sum_{v\in[n]} d_{G_n}(v)$.

$m(G_n) \geq 1$ for $(G_n, o_n) \in \mathcal{G}_*$ (assume (G_n, o_n) has no isolated nodes)

Define $G_n^* \in \mathcal{G}_* \ni$

$$\mathbb{P}(G_n^* = (G_n, v)) = \frac{d_{G_n}(v)}{m(G_n)} \mathbb{P}((G_n, o_n) = (G_n, v)) = \frac{d_{G_n}(v) \setminus n}{m(G_n)}$$

Biasing the prob. measure by the degree.

$$\mathbb{P}((G_n, o_n) = \cdot) \leq m(G_n)\, \mathbb{P}(G_n^* = \cdot) \leq f(o)\, \mathbb{P}(G_n^* = \cdot) \quad -(2)$$

$\Rightarrow$ if $(G_n^*)_{n \geq 1}$ is tight then so is (G_n, o_n).

Further $\mathbb{P}(o_n^* = v) = \dfrac{d_{G_n}(v)}{n\, m(G_n)} = \dfrac{d_{G_n}(v)}{\sum_{u \in [n]} d_{G_n}(u)}$

$$\Rightarrow \mathbb{E}\left[d_{o_n^*}^{(n)} \mathbb{1}[d_{o_n^*} \geq K]\right] = \sum_{v \in [n]} d_{G_n}(v) \mathbb{1}[d_{G_n}(v) \geq K]\, \mathbb{P}[o_n^* = v]$$

$$= \frac{1}{n} \sum_{v \in [n]} \frac{d_{G_n}(v)^2 \mathbb{1}[d_{G_n}(v) \geq K]}{m(G_n)}$$

$$= \frac{\mathbb{E}\left[(d_{o_n}^{(n)})^2 \mathbb{1}[d_{o_n}^{(n)} \geq K]\right]}{\mathbb{E}[d_{o_n}^{(n)}]}$$

(Ex) $\Rightarrow d_{o_n^*}^{(n)}$ is U.I.

From (2) ETPT, G_n^* is tight. T.S.T, G_n^* satisfies (1)

$\mathbb{P}(O_n^* = \cdot)$ – Stationary measure for SRW on G_n.

$\Rightarrow X_n =$ Unif. neighbour of $O_n^* \stackrel{d}{=} O_n^*$

$$\mathbb{P}(X_n = v) = \sum_{u \in N_v} \underbrace{\frac{d_{G_n}(u)}{n\, m(G_n)}}_{= \mathbb{P}(o_n^* = u)} \times \underbrace{\frac{1}{d_{G_n}(u)}}_{= \mathbb{P}(X_n = v \mid o_n^* = u)} = \frac{d_{G_n}(v)}{n\, m(G_n)}$$

$$\Rightarrow (G_n, o_n^*) \stackrel{d}{=} (G_n, X_n)$$

$$\mathbb{P}((G_n, o_n^*) \in F_{r+1}^M) \leq \mathbb{P}(d_{o_n^*}^{(n)} > d) + \mathbb{E}\left[\mathbb{1}[d_{o_n^*}^{(n)} < d]\, \mathbb{P}[(G_n, o_n^*) \in F_{r+1}^M \mid d_{o_n}^{(n)}]\right]$$

By u.I., $\exists d'$ $\ni \forall n$

$$\leq \frac{\varepsilon}{2} + \mathbb{E}\left[d\, \mathbb{1}[d_{o_n^*}^{(n)} < d]\, \mathbb{P}((G_n, X_n) \in F_r^M \mid d_{o_n^*}^{(n)})\right]$$

$\Big[(G_n, o_n^*) \in F_{r+1}^M \implies d_{G_n}(v) > M$ for some $v \in B_{r+1}(o_n^*)$

$\implies d_{G_n}(v) > M$ for some $v \in B_r(u)$ where $u \sim o_n^*$ $\Big]$

$$\leq \frac{\varepsilon}{2} + d\, \mathbb{P}((G_n, X_n) \in F_r^M)$$

$$= \frac{\varepsilon}{2} + d\, \mathbb{P}((G_n, o_n^*) \in F_r^M) \qquad \left[(G_n, X_n) \overset{d}{=} (G_n, o_n^*)\right]$$

Proof complete by induction.

A family of random variables $\{X_i\}_{i\in I}$ is called *uniformly integrable (u.i.)* if for any given $\epsilon > 0$, there exists A large enough so that

$$\sup_{i\in I} \mathbb{E}\left(|X_i| \mathbf{1}[|X_i| > A]\right) < \epsilon.$$

Show that $\{X_i\}_{i\in I}$ is u.i. iff $\sup_{i\in I} \mathbb{E}[|X_i|] < \infty$ (i.e., uniformly bounded in L^1) and also that for any $\epsilon > 0$, there exists $\delta > 0$ such that for any measurable A with $\mathbb{P}(A) < \delta$ implies that $\sup_{i\in I} \mathbb{E}[|X_i| \mathbf{1}_A]] < \epsilon$.

If degree of the root has bounded pth moments for p > 1 then it is uniformly integrable.

$$\mathbb{E}\left[|X|\, \mathbb{1}[|X| \geq A]\right] \leq \mathbb{E}[|X|^p]^{1/p}\, \mathbb{P}(|X| \geq A)^{1/q}$$

$$\frac{1}{p} + \frac{1}{q} = 1.$$

- If $d_{o_n}^{(n)}$ has bdd p^{th} moments for some $p > 1$,

then (G_n, o_n) is tight i.e., has convergent subsequences.

$$\mathbb{E}[(d_{o_n}^{(n)})^p] = \sum k^p\, p_{G_n}(k), \quad p_{G_n}(k) = \frac{\#\{v : d_v = k\}}{n}$$

X_n — $p_{G_n}(k)$: $\rightarrow$ lmf.

often $p_{G_n}(k) \to p_G(k)$ & $\mathbb{E}[X_n^p] \to \sum k^p p_G(k)$.

Defn: (estimable functions) Let $\mathcal{G}$ be the set of fin. unlabeled graphs & $\mathcal{H} \subseteq \mathcal{G}$. A fn $f: \mathcal{G} \to \mathbb{R}$ is ESTIMABLE over $\mathcal{H}$ if for any $G_n \in \mathcal{H}$ $\ni G_n \xrightarrow{LWC} (G, o)$ we have that $f(G_n) \to \mathbb{E}\hat{f}(G, o)$ for some $\hat{f}: \mathcal{G}_* \to \mathbb{R}$.

Eg. $\varphi: \mathcal{G}_* \to \mathbb{R}$ bdd cts.

& $$f(G) = \frac{1}{|V|} \sum_{v \in V} \varphi(G, v)$$

$$f(G_n) = \mathbb{E}\varphi(G_n, o_n) \to \mathbb{E}\varphi(G, o)$$

$\Rightarrow$ f is estimable over $\mathcal{G}$

Eg. $$f(G) = \frac{1}{|V|} \sum_{\substack{S \subseteq V \\ |S| = p}} \mathbb{1}\{G(S) \cong H\},$$ H - transitive graphs on p vertices.

$G(S)$ - G restricted to the subset S.

$$= \frac{1}{p|V|} \sum_{v \in V} \varphi(G, v)$$

$\varphi(G, v)$ = # of subgraphs rooted at v & isomorphic to H^* where $H^* = (H, o)$ for an arbitrary vertex o

[Since H is transitive, we can choose any vertex $o \in H$ as its root].

= # of subgraphs isomorphic to H & containing v.

If H is connected then $\varphi(G, v)$ is cts. (Ex)

$$f(G_n) = \frac{1}{p}\mathbb{E}[\varphi(G_n, o_n)]$$

(Ex). $$\to \frac{1}{p}\mathbb{E}[\varphi(G, o)]$$ if $\{\varphi(G_n, o_n)\}_{n \geq 1}$ is U.I.

$\mathcal{G}_d = \{ G \in \mathcal{G} : \max_{v \in V} d_G(v) \leq d \}$

$\Rightarrow \varphi(G, v)$ is bounded $\forall\, G \in \mathcal{G}_d$

$\Rightarrow \quad f(G_n) \longrightarrow \frac{1}{p} \mathbb{E}[\varphi(G, 0)] \quad \forall\, G_n \subset \mathcal{G}_d.$

$\Rightarrow f(G_n)$ is estimable over $\mathcal{G}_d$.

<u>Eg.</u> $f(G) = \frac{1}{|V|} \log c_k(G)$, $\quad c_k(G) = \#$ proper k-colorings of G

is estimable over $\mathcal{G}_d$. $\quad$ with $k > 2d$.

$f(G) = \frac{1}{|V|} \log t(G) \qquad t(G) = \#$ spanning trees of G.

is estimable over conn. graphs.

<u>Ex:</u> What if H isn't transitive?

Sparse Random Graphs : Assignment 3

Yogeshwaran D.

November 2, 2020

Submit solutions via Moodle by 15th November 10:00 PM.

1. Let G_n converge in probability in the local weak sense to (G, o). Let $(o_n^{(1)}, o_n^{(2)})$ be two independent uniformly chosen vertices in $[n]$. Show that $((G_n, o_n^{(1)}), (G_n, o_n^{(2)})) \xrightarrow{d} ((G, o), (G', o'))$ where (G', o') is an independent copy of (G, o).

2. Construct an example where G_n converges in probability in the local weak sense to (G, o), while $\frac{|C_{max}|}{n} \xrightarrow{p} \eta < \zeta = \mathbb{P}(|C(o)| = \infty)$.

3. Let G_n be a finite (possibly disconnected) random graph and converge in probability in the local weak sense to (G, o). Let $\zeta = \mathbb{P}(|C(o)| = \infty)$. Assume that
$$\limsup_{k\to\infty} \limsup_{n\to\infty} n^{-2}\mathbb{E}[|\{x, y \in [n] : |C(x)|, |C(y)| \geq k, x \notin C(y)\}|] > 0.$$
Then prove that for some $\epsilon > 0$,
$$\limsup_{n\to\infty} \mathbb{P}(|C_{max}| \leq n(\zeta - \epsilon)) > 0.$$

4. Under the assumptions of Q.3., show that for some $\epsilon > 0$,
$$\limsup_{n\to\infty} \mathbb{P}(|C_2| \geq \epsilon n) > 0,$$
where C_2 is the second largest component with ties broken arbitrarily.

5. The *size-biased version* X^* of a non-negative random variable X is defined as
$$\mathbb{P}(X^* \leq X) = \frac{\mathbb{E}[X 1[X \leq x]}{\mathbb{E}[X]}.$$
Show that when $(d_n^{o_n})_{n\geq 1}$ forms a uniformly integrable sequence of random variables, there exists a subsequence along which D_n^*, the size-biased version of $D_n = d_n^{o_n}$, converges in distribution. Here G_n is a sequence of

6. Let $G_n = G(n, p)$ for $p = \lambda/n$. Using direct computations (i.e., not using local weak convergence), show that
$$\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}(d_{G_n}(o_n^{(1)}, o_n^{(2)}) \leq \frac{\log n}{\log \lambda} - K) = 0,$$
where $o_n^{(1)}, o_n^{(2)}$ are independent uniformly chosen vertices in $[n]$.

10/11/20: L10 - UNIMODULARITY

$G_n \xrightarrow{LW-P} (G, o)$. G_n seq. of finite (random) graphs.
What can we say about set of limit points (G, o)?
What about the root in relation to the graph?
Root has to be "the typical" vertex of the graph.
What is a typical vertex?
Given a finite graph, typical vertex = uniformly chosen?
What abt. infinite graphs?

<u>PROPOSITION</u>: A finite rooted random graph (G, o) has root uniformly distributed iff

$$\mathbb{E}\Big[\sum_{v \in V} \underbrace{f(G, o, v)}_{\text{doubly rooted.}}\Big] = \mathbb{E}\Big[\sum_{v \in V} f(G, v, o)\Big] \quad - (1)$$

$\forall$ non-neg. fns $f(G, u, v)$ of the graph & $u, v \in V$.

Proof: o - Unif rooted.

$$\mathbb{E}\Big[\sum_{v \in V} f(G, o, v)\Big] = \sum_{v \in V} \mathbb{E}\Big[\frac{1}{|V|} \sum_{u \in V} f(G, u, v)\Big]$$

$$\text{(Fubini-Tonelli)} = \sum_{u \in V} \mathbb{E}\Big[\frac{1}{|V|} \sum_{v \in V} f(G, u, v)\Big]$$

$$= \sum_{u \in V} \mathbb{E}\big[f(G, u, o)\big]$$

$$= \mathbb{E}\left[\sum_{u \in V} f(G, u, 0)\right]$$

$\Rightarrow$ (1).

Converse: Define $f(G, x, y) = \dfrac{h(G, x)}{|V|}$

$h : \mathcal{G}^* \longrightarrow \mathbb{R}_+$.

$$\mathbb{E}\left[\sum_{v \in V} f(G, 0, v)\right] = \mathbb{E}\left[h(G, 0)\right]$$

$$\parallel$$

$$\mathbb{E}\left[\sum_{v \in V} f(G, v, 0)\right] = \mathbb{E}\left[\sum_{v \in V} \frac{h(G, v)}{|V|}\right]$$

$$\Rightarrow \mathbb{E}\left[h(G, 0)\right] = \frac{1}{|V|} \sum_{v \in V} \mathbb{E}\left[h(G, v)\right]$$

$\Rightarrow 0 \overset{d}{=} \mathrm{Unif}(V)$. ∎

Interpretation of (1)

$f(G, u, v)$ — Mass sent from u to v in G.

$\sum_{v \in V} f(G, u, v)$ — Total outgoing mass at u

$\sum_{v \in V} f(G, v, u)$ — incoming — at u

MTP — Mass transport principle (i.e., (1)).

(Ignores graph structure).

$\Rightarrow$ $\mathbb{E}$[outgoing mass at root]
$= \mathbb{E}$[incoming mass at root].

Uniform root $\equiv$ MTP
in fin. graphs

∞ graphs ?

$\mathcal{G}_{**}$ - space of doubly rooted loc. finite graphs / $\cong$

$(G, u, v) \cong (G', u', v')$ if
$\exists$ a graph isomorphism $\phi : G \to G'$
$\ni \phi(u) = u'$, $\phi(v) = v'$.

So $f : \mathcal{G}_{**} \to \mathbb{R}$ $\Rightarrow$ $f(G, u, v) = f(G', u', v')$
if $(G, u, v) \cong (G', u', v')$.

DEFN $(G, o) \in \mathcal{G}^*$, random graph. We say (G, o) satisfies MTP if

$$\mathbb{E}\Big[\sum_{v \in V} f(G, o, v)\Big] = \mathbb{E}\Big[\sum_{v \in V} f(G, v, o)\Big].$$

$\forall$ $f : \mathcal{G}_{**} \to \mathbb{R}_+$.

(G, o) called unimodular if (G, o) satisfies MTP.

$\to$ Restriction to $\mathcal{G}_{**}$ is necessary (Ex.)

$\to$ finite (G, o) is unimodular if o is uniform.

PROPN: $(G,o) \in \mathcal{G}^*$ random graph is unimodular iff it satisfies MTP $\forall f : \mathcal{G}_{**} \to \mathbb{R}_+$ $\ni f(G, x, y) = 0$ if $x \not\sim y$ in G.

[INVOLUTION INVARIANT].

Proof later. i.e., check MTP on fns that are non-trivial only on edges & you get full MTP.

Eg: (1) $(\overset{G_n}{G(n,p)}, 1)$ is unimodular.

$$\Longleftrightarrow \mathbb{E}\Big[\sum_{i \neq 1} h(G_n, 1, i)\Big] = \mathbb{E}\Big[\sum_{i \neq 1} h(G_n, i, 1)\Big]$$

$\forall h : \mathcal{G}_{**} \to \mathbb{R}_+$. (check)

Let G be a Cayley graph. Is (G,o) unimodular?

G is transitive if $(G,u) \cong (G,v)$ $\forall u, v \in V$.

G is symmetric if $(G,u,v) \cong (G,u',v')$

$\forall u, v, u', v' \in V \ni u \sim v$ & $u' \sim v'$.

A connected symmetric graph is transitive. (Prove!)

Propn: A conn. symmetric graph G with an arbitrary root is unimodular.

why?

Proof:

For $o \sim v$, $f(G, o, v) = f(G, v, o)$ by symmetry & $f : \mathcal{G}_{**} \to \mathbb{R}$. — (2)

Set $f : \mathcal{G}_{**} \to \mathbb{R} \ni f(G, u, v) = 0$ if $u \not\sim v$.

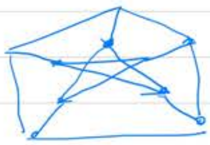
(2) $\Rightarrow$ if $o \sim v$ $f(G, o, v) = f(G, v, o)$

$$\Rightarrow \sum_{v \in V} f(G,o,v) = \sum_{v \in V} f(G,v,o)$$

By Propn on II, (G,o) is unimodular. ☒

→ Cayley graphs are conn. & transitive. Symmetric ? Unimodular ? (see later).

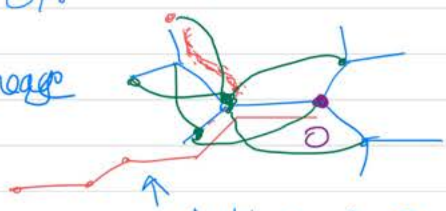
→ Reg. trees are conn. & symmetric.



– Peterson graph ✓ Smallest non-Cayley graph.

Eg: (Grandfather graph) G.

we can get ancestral line/ lineage for all vertices.

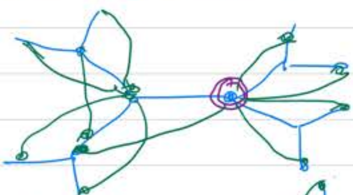


↑ ancestral line of o. any unbdd path from o.

⇒ Every vertex has a lineage & in particular a grandfather.

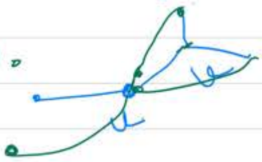
Connect each vertex to its grandfather.

G is transitive. (check)



G isn't symmetric.

$$(G,u,v) \not\cong (G,v,u).$$



$$f(G,u,v) = \mathbb{1}[u \text{ is grandfather of } v]$$

$$\underbrace{\sum_v f(G,u,v)}_{\#\text{ of grand-children of } u} = 4 \neq \underbrace{\sum_v f(G,v,u)}_{\#\text{ of grand fathers of } u} = 1.$$

$\Rightarrow (G,u)$ isn't unimodular for any $u \in V$.

Defn: $\mathcal{G}^f = \{ G : |G| < \infty \} \cong \{ (G,o) : G \in \mathcal{G}_f,\ o \text{ unif} \} \subseteq \mathcal{G}_*$.

$\mathcal{G}_s$ = "Closure of $\mathcal{G}^f$ under LWC"

$= \{ (G,o) : G_n \xrightarrow{LW\text{-}d} (G,o) \ \& \ |G_n| < \infty \} \subseteq \mathcal{G}_*$

Clarification: $\mathcal{P}_*$ — prob. measures on $\mathcal{G}_*$ i.e., rooted random graphs.

Sofic prob. measures $\mathcal{P}_s$ — Prob. measures on $\mathcal{G}_*$ obtained as a limit of unif. rooted finite graphs.

We have slightly abused notation by representing prob. measures by random elements.

A rooted (random) graph (G,o) is SOFIC if $(G,o) \in \mathcal{P}_s$.

Qn: Is $\mathcal{P}_s = \mathcal{P}_*$?

THM: Any Sofic graph is unimodular.
i.e., Unimodular random graphs are closed under LWC.

$\Rightarrow \mathcal{P}_s \subseteq \mathcal{P}_u \subsetneq \mathcal{P}_*$. $\mathcal{G}^f \subseteq \mathcal{P}_u$.

OPEN QUESTIONS — Is $\mathcal{P}_s = \mathcal{P}_u$?

EXTRAS.

Suppose P is a group property.
Then we can say a graph G has property P
if $Aut(G)$ has property P
Gives a way to translate group-theoretic prop. to Graph properties.

Ref: Group invariant percoln on graphs
— Benjamini, Lyons, Peres, Schramm.
Processes on Unimodular random networks
— Aldous & Steele.

Application of THM: Amenable Cayley graphs with arbitrary root are sofic.

Ref for unimodularity: Blaszczyszyn Ch. 5 & Bordenave - Counting.....

16/11 : L11 : SOFICITY & UNIMODULARITY.

$\mathcal{P}_f := \{ (G, o) : G \text{ finite}, \; o \stackrel{d}{=} \mathrm{Unif}([v]) \}$

$\mathcal{P}_f \subseteq \mathcal{P}_u$;

THM: $\mathcal{P}_u$ is closed under local weak convergence.

$\Rightarrow \mathcal{P}_s = \overline{\mathcal{P}_f} \subseteq \mathcal{P}_u$.

Proof: Let $(G_n, o_n) \in \mathcal{P}_u$ $\ni$ $(G_n, o_n) \xrightarrow{LW\text{-}d} (G, o)$.

Let $f : \mathcal{G}_{**} \to \mathbb{R}_+$.

Let $t > 0$ & $g \in \mathcal{G}_*$ $\ni$ $g \subseteq B_r^{(g)}(o)$.

$$f_{t,g}(G,u,v) = t \wedge f(G,u,v) \mathbb{1}[d_G(u,v) \le t] \times \mathbb{1}[(G,u)_t \cong g].$$

$$\Rightarrow \overline{f}_{t,g}(G,u) = \sum_{v \in V_n} f_{t,g}(G,u,v)$$

$$= \sum_{v \in B_t^G(u)} [t \wedge f(G,u,v)] \mathbb{1}[(G,u)_t \cong g]$$

$$\underline{f}_{t,g}(G,u) = \sum_{v \in V_n} f_{t,g}(G,v,u)$$

$$= \sum_{v \in B_t^G(u)} t \wedge f(G,v,u) \mathbb{1}[(G,v)_t \cong g]$$

$\bar{f}_{t,g}$ is bdd & depends on t-nbhd of root.

$\underline{f}_{t,g}$ is also bdd & depends on $2t$-nbhd of root. ??

$\Rightarrow$ $\bar{f}_{t,g}$, $\underline{f}_{t,g}$ are bdd cts fnds on $\mathcal{G}_*$.

$$\Rightarrow \quad \mathbb{E}[\bar{f}_{t,g}(G_n,o_n)] \overset{\text{unimodularity of } (G_n, o_n)}{=} \mathbb{E}[\underline{f}_{t,g}(G_n,o_n)]$$

(by LWC) $\longrightarrow$ $\downarrow$ $\qquad\qquad\downarrow$

$$\mathbb{E}[\bar{f}_{t,g}(G,o)] = \mathbb{E}[\underline{f}_{t,g}(G,o)]$$

$\Rightarrow$ (G,o) satisfies MTP for $f_{t,g}$.

Summing over g & Fubini-Tonelli,

(G,o) satisfies MTP for $f_t(G,u,v) = (t \wedge f(G,u,v)) \times \mathbb{1}[d_G(u,v) \le t]$

$$\Rightarrow \quad \mathbb{E}\Big[\sum_{v \in V} f_t(G,o,v)\Big] = \mathbb{E}\Big[\sum_{v\in V} f_t(G,v,o)\Big]$$

$\forall\, v \in V$, $f_t(G,u,v) \uparrow f(G,u,v)$ as $t \to \infty$

By MCT,

$$\mathbb{E}\Big[\sum_{v \in V} f_t(G,o,v)\Big] = \mathbb{E}\Big[\sum_{v\in V} f_t(G,v,o)\Big]$$

$\downarrow t\to\infty \qquad\qquad \downarrow t \to \infty$

$$\mathbb{E}\Big[\sum_{v \in V} f(G,o,v)\Big] = \mathbb{E}\Big[\sum_{v\in V} f(G,v,o)\Big] \qquad \blacksquare$$

$\mathcal{P}_u(\mathcal{T})$ = Unimodular random graphs supported on rooted trees

$\mathcal{P}_u(\mathcal{T}) \subseteq \mathcal{P}_s$ Elek, Bowen

See Bordenave : Counting & optimizing ....

$\mathcal{P}_s = \mathcal{P}_u$?

<u>Eg</u>: (1) Amenable Cayley graphs are sofic.

$B_r^G(o) \xrightarrow{LW-C} (G, o)$ for amenable Cayley graphs

[Assignment 4].

(2) BGW(λ) is sofic & so unimodular.

(3) (Ex.) G - Amenable Cayley graph.

$G_p = (V, E_p) \subseteq G$

$E_p = \{e \in E : X_e = 1\}$ - Bond percolation.

$X_e, e \in E$ i.i.d. Ber(p).

Is $(G_p, 0)$ unimodular ?

(4) Cayley graphs: $f : \mathcal{G}_{**} \to \mathbb{R}_+$ G- Cayley graph

$\Rightarrow f(G, u, v) = f(G, \gamma u, \gamma v), \gamma \in G.$

$\hookrightarrow$ invariant by left-multiplication

as $(G, u, v) \cong (G, \gamma u, \gamma v), \gamma \in G.$

$$\sum_{v \in G} f(G,0,v) = \sum_{v \in G} f(G,0,v^{-1}) \quad , \quad 0 - \text{identity}.$$

$$\left(\substack{\text{left multiply} \\ \text{by } v}\right) \quad = \sum_{v \in G} f(G,v,0)$$

$\Rightarrow (G,0)$ is unimodular.

THM: (Induction invariance):

Let $(G,0) \in \mathcal{G}^*$ $\ni$ MTP holds for all $f: \mathcal{G}_{**} \to \mathbb{R}_+ \ni f(G,u,v) = 0$ if $u \not\sim v$.

The $(G,0)$ is unimodular.

Proof: $f: \mathcal{G}_{**} \to \mathbb{R}_+$ can be written as

$$f(G,u,v) = \sum_{t=0}^{\infty} f_t(G,u,v)$$

where $f_t(G,u,v) = 0$ if $d(u,v) \neq t$.

Assumption is MTP for f_1.

Set $f = f_t$ for some $t \geq 2$. We'll prove by induction.

$\forall\, k \geq 1 \quad \partial B_k^G(u) = B_k^G(u) \setminus B_{k-1}^G(u)$

If $x \in \partial B_t^G(u)$, let $\pi(G,u,x) = \#$ geodesic paths from u to x ≥ 1

geodesic = shortest paths = paths of length $d(u,x)$.

For $y \in \partial B_{t-1}^G(u)$, $\pi(G,u,x,y) = \#$ geodesic paths from u to x that hit y first.

$$\pi(G,u,x) = \sum_{y \in \partial B_{t-1}^G}^{k} \pi(G,u,x,y) \quad — \quad (1)$$

$$\forall\, y \in \partial B_{t-1}^G(u), \quad h(G,u,y) = \sum_{x \in \partial B_t^G(u)} f(G,u,x) \frac{\pi(G,u,x,y)}{\pi(G,u,x)}$$

$\& \ h(G,u,y) = 0 \quad \forall\, y \notin \partial B^G_{t-1}(u).$

$$\sum_{v \in V} h(G,u,v) = \sum_{y \in \partial B^G_{t-1}(u)} \sum_{x \in \partial B^G_t(u)} f(G,u,x) \frac{\pi(G,u,x,y)}{\pi(G,u,x)}$$

$$\text{(interchange sums)} \quad = \sum_{x \in \partial B^G_t(u)} \sum_{y \in \partial B^G_{t-1}(u)} f(G,u,x) \frac{\pi(G,u,x,y)}{\pi(G,u,x)}$$

$$\text{(by ①)} \quad = \sum_{x \in \partial B^G_t(u)} f(G,u,x)$$

$$= \sum_{v \in V} f(G,u,v).$$

By inductive assumption, h satisfies MTP & now so does f. ∎

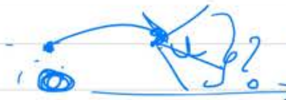
We know $BGW(\lambda)$ is unimodular.

Ex. S.T. $BGW(P)$ is unimodular iff P is $Poisson(\lambda)$ distribn.

Let P be pmf on $\mathbb{Z}_+$, the offspring distribn & $\sum l P(l) < \infty$

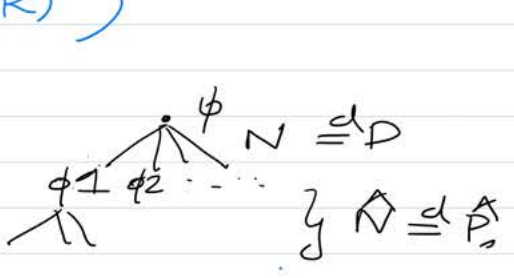
let $\hat{P}(k-1) = \dfrac{k P(k)}{\sum_{l \geq 0} l P(l)}$ be the size-biased distribn of P.

Ex. when is $\hat{P} = P$?

If degrees are i.i.d. P,  $deg_u - 1 \overset{d}{=}$?

(equivalently, $\dfrac{\#\{u : deg_u = k\}}{|V|} = P(k)$)

Defn: $UGW(P)$ is the random tree $\ni N_\phi \overset{d}{=} P$, $N_i \overset{d}{=} \hat{P}$ $\forall\, i \neq \phi$. & N_ϕ, N_i are indep.



THM: $UGW(P)$ is a unimodular random graph.

Eg: $P(k) = \delta_d(k) = \mathbb{1}[k=d]$, $d \geq 2$.

$$\hat{P}(k-1) = \frac{k P(k)}{d} = \frac{d\, \mathbb{1}[k=d]}{d} = \mathbb{1}[k-1 = d-1].$$

$UGW(P)$ is d-regular tree.

$\Rightarrow$ d-regular trees are unimodular.

Eg. UGW(p) are sofic limits of configuration models.

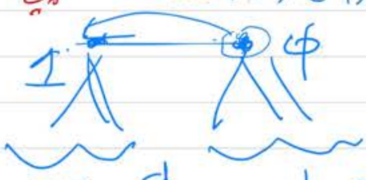
Proof:

EFPT MTP for $f \ni f(G,u,v) = 0$ if $u \nsim v$.

$$\mathbb{E}\Big[\sum_{i=1}^{N_\phi} f(T,\phi,i)\Big] \underset{(?)}{=} \sum_{k=1}^{\infty} k\, p(k)\, \mathbb{E}\big[f(T,\phi,1) \mid N_\phi = k\big]$$

$$= \underline{\mathbb{E}N} \sum_{k=0}^{\infty} \hat{p}(k)\, \mathbb{E}\big[f(T,\phi,1) \mid \hat{N}_\phi = k+1\big]$$

?? ← $f((T',\phi),(T_1,1))$



$T_1 \overset{d}{=} T' \overset{d}{=} GWT(\hat{p})$

T_1 indep. of T'

$X \overset{d}{=} X'$ rand. elements
X, X' indep. $\mathbb{E} f(X,X') = \mathbb{E} f(X',X)$

24/11/20: L12 - DOUBLY ROOTED & WEIGHTED GRAPHS

Unimodularity of UGW(P), please see Bordenave (Random graphs & Comb. optimization) Lemma 3.10.

Proof uses change of measure from P to $\hat{P}$ and symmetry under $\hat{P}$.

$\mathcal{G}_{**}$ - Doubly rooted graphs

$(G,u,v) \cong (G',u',v')$ if $G \cong G'$ with $u \mapsto u'$, $v \mapsto v'$.

$$B_t^G(u;v) = B_t^G(u) \cup B_t^G(v)$$

$=$ induced subgraph on $\{w : \min\{d_G(u,w), d_G(v,w)\} \le t\}$

We define metric on $\mathcal{G}_{**}$ similar to the local weak metric on $\mathcal{G}_*$.

$$R(G,G') = \sup\{r \in \mathbb{N} : B_r^G(u,v) \cong B_r^{G'}(u',v')\}$$

$$d_*((G,u,v),(G',u',v')) = \frac{1}{R(G,G')+1}$$

Ex: S.T. $(\mathcal{G}_{**}, d_*)$ is a complete separable metric space.
(A)

Ex: S.T. π $(G,u,v) \longmapsto (G,u)$ is continuous.
(A)

THM: $\mathcal{P}_u$ is closed under local weak convergence.

$\Rightarrow \mathcal{P}_s = \overline{\mathcal{P}_{f}} \subseteq \mathcal{P}_u$.

Proof: If MTP holds for $f_t : \mathcal{G}_{**} \to \mathbb{R}_+$

$\ni f_t(G, u, v) = 0$ iff $d_G(u,v) > t$ (with $t > 0$)

then MTP holds, $\forall f$. (arbitrary $f = \sum_{t=0}^{\infty} f_t \cdots$)

Fix $t > 0$ & $f \ni f(G,u,v) = 0$ if $d_G(u,v) > t$.

$\exists$ simple functions $f_n \ni f_n \uparrow f$ & $f_n = 0$ if $d_G(u,v) > t$.

Simple function f is of form

$$f([G,u,v]) = \sum_{l=1}^{L} \alpha_l \mathbb{1}[[G,u,v] \in A_l], \quad A_l \in \mathcal{B}(\mathcal{G}_{**})$$
$$\times \mathbb{1}[d_G(u,v) \le t] \qquad \alpha_l \geq 0.$$

By linearity, it is enough to prove MTP

for $f([G,u,v]) = \mathbb{1}[[G,u,v] \in A] \mathbb{1}[d_G(u,v) \le t]$

$A \in \mathcal{B}(\mathcal{G}_{**})$

$\mathcal{A} = \{A \in \mathcal{B}(\mathcal{G}_{**}) : f([G,u,v])$ as above satisfies MTP $\}$

$\mathcal{A}$ is closed under countable union & complements.

(by MCT)

Also trivially true for $A = \mathcal{G}_{**}$.

So $\mathbb{1}[G \in A^c] \mathbb{1}[d_G(u,v) \le t] = \mathbb{1}[d_G(u,v) \le t]$

$- \mathbb{1}[G \in A] \mathbb{1}[d_G(u,v) \le t]$

So enough to prove MTP for A in a generating class. $\mathcal{A}_*$.

Define $\mathcal{A}_* = \{ A_{s,(H,i,j)} : s > 0, (H,i,j)$ is a finite s-depth graph $\ni d_H(i,j) \le t \}$

$(H,i,j) \cong B_s^H(i,j)$.

$$A_{s,(H,i,j)} := \{ (G,u,v) : B_s^G(u,v) \cong (H,i,j) \}$$

Since $\sigma(\mathcal{A}_*) \overset{\text{(check)}}{=} \mathcal{B}(\mathcal{G}_{**})$ & so ETPT $\mathcal{A}_* \subseteq \mathcal{A}$.

Consider $A_{s,(H,i,j)}$

$$\text{Set } \underline{h}((G,u)) = \sum_{v \in B_t^G(u)} \mathbb{1}[B_s^G(u,v) \cong (H,i,j)]$$

$$= \sum_{v \in V} \mathbb{1}[B_s^G(u,v) \cong (H,i,j)] \times \mathbb{1}[d_G(u,v) \le t]$$

$$= \sum_{v \in V} \mathbb{1}[(G,u,v) \in A_{s,(H,i,j)}] \mathbb{1}[d_G(u,v) \le t]$$

$$\bar{h}((G,u)) = \sum_{v \in B_t^G(u)} \mathbb{1}[B_s^G(v,u) \cong (H,i,j)]$$

$\Rightarrow$ $\bar{h}$ & $\underline{h}$ are cts fns on $\mathcal{G}_*$ as they depend on $B_{s+t}^G(u)$.

$\Rightarrow$ Suppose $s > t$ & $\underline{h}((G,u)) > 0$.

Then $B_t^G(u) \cong B_t^H(i)$

$\Rightarrow$ $\underline{h}((G,u)) \le |B_t^H(i)|$

llly $\overline{h}((G,u)) \le |B_t^H(j)|$

Thus $\underline{h}$ & $\overline{h}$ are bdd cts fnls on $\mathcal{G}_*$.

Now $(G_n,O_n) \xrightarrow{Lw\text{-}d} (G,O)$.

so $\mathbb{E}\,\underline{h}(G_n,O_n) \longrightarrow \mathbb{E}\,\underline{h}(G,O)$.

& $\mathbb{E}\,\overline{h}(G_n,O_n) \longrightarrow \mathbb{E}\,\overline{h}(G,O)$.

By assumption (G_n,O_n) is unimodular & so

Ex: Prove via Lusin's thm.

$$\mathbb{E}\,\underline{h}(G_n,O_n) = \mathbb{E}\,\overline{h}(G_n,O_n)$$

$$\Rightarrow \mathbb{E}\,\underline{h}(G,O) = \mathbb{E}\,\overline{h}(G,O)$$

$\Rightarrow$ $\mathcal{A}_* \subseteq \mathcal{A}$ as required. ∎

$\mathcal{G}_E$ - Rooted graphs with edge weights.

A $\mathbb{R}$-weighted graph (G,w) is a graph $G=(V,E)$ with weight fn $w: V^2 \to \mathbb{R}$ $\ni$ $w(u,v)=0$ if $(u,v) \notin E$.

w is EDGE-SYMMETRIC if $w(u,v) = w(v,u)$

& $w(u,u)=0$.

(G,w) is LOCALLY FINITE if $\forall\, v \in V$

$$\sum_{u \in V} \left(|w(u,v)| \vee |w(v,u)|\right) \mathbb{1}[(u,v)\in E] < \infty.$$

G loc. fin $\Rightarrow$ (G,w) is loc. fin.

A network is (G,ω) where $G=(V,E)$ is a loc. fin. graph & $\omega : (V \cup E) \to \Omega$, (Polish space).
Mark space

$\boxed{\text{DEFN}}$ $(G,\omega) \cong (G',\omega')$ if $\exists$ a graph isomorphism $\phi : G \to G'$ $\ni$ $\omega'(\phi(v)) = \omega(v)$, $\omega'(\phi(e)) = \omega(e)$ $\forall$ $v \in V$ & $e \in E$. [Network isomorphism]

$(G,o,\omega) \cong (G',o',\omega')$ if $\phi : (G,\omega) \to (G',\omega')$ isomorphism s.t. $\phi(o) = o'$. [Rooted network isomorphism]

$\underline{\mathcal{G}_*(\Omega)} := \{ \underbrace{[G,o,\omega]}_{\text{eq. class.}} : (G,o,\omega) \text{ rooted network} \}$

$g_1, g_2 \in \mathcal{G}_*(\Omega)$ $\quad g_i = (g_i, o_i, \omega_i)$

$$d(g_1,g_2) = \frac{1}{1+T}$$

where $T = \sup\{ t>0 : \exists$ rooted isomorphism $\phi : B_t^{g_1}(o_1) \to B_t^{g_2}(o_2) \ni d_\Omega(\omega_1(v), \omega_2(\phi(v))) \le 1/t$
& $d_\Omega(\omega_1(e), \omega_2(\phi(e))) \le 1/t$
$\forall v, e \in B_t^{g_1}(o_1) \}$.

Ex: $\mathcal{G}_*(\Omega)$ is a seperable & complete m. space.
(A)

Ex: Let $\psi : \mathcal{G}_* \to \Omega$, $\phi : \mathcal{G}_{**} \to \Omega$ m'ble.
(A)
Define (G,ω) as $\omega(u) = \psi(G,u)$
$\omega(u,v) = \phi(G,u,v)$.

If (G,o) is unimodular, so is (G,o,ω).

$\mathcal{G}_{**}(\Omega) = \{ [G,u,v,\omega] : (G,u,v,\omega)$ - doubly rooted network $\}$

$d(g_1,g_2) = \frac{1}{1+T}$ $\qquad g_i = (g_i, u_i, v_i, w_i)$

$T = \sup\{ t>0 : B_t^{g_1}(u_1,v_1) \overset{\phi}{\cong} B_t^{g_2}(u_2,v_2) \ \exists$
$d_\Omega(\omega_1(v), \omega_2(\phi(v))) \le 1/t, \quad \forall v \in B_t^{g_1}(u_1,v_1)$
$d_\Omega(\omega_1(e), \omega_2(\phi(e))) \le 1/t \ \forall e \in B_t^{g_1}(u_1,v_1) \}$

(G,o,ω) is unimodular if $\forall f : \mathcal{G}_{**}(\Omega) \to \mathbb{R}_+$

$$\mathbb{E}\Big[\sum_{v\in V} f(G,o,v,\omega) \Big] = \mathbb{E}\Big[\sum_{v \in V} f(G,v,o,\omega) \Big].$$

Ex (A) Percolation preserves unimodularity.

Let (G,o,ω) be a unimodular random network.
Let $B \in \mathcal{B}(\Omega)$

Define $\hat{G} = (V(G), \hat{E} = \{ (u,v) \in E : \omega(u,v) \in B, \omega(v,u) \in B \})$

$\hat{\omega}(u,v) = \omega(u,v)\, \mathbb{1}[\omega(u,v) \in B]\, \mathbb{1}[\omega(v,u) \in B]$.

S.T. $(\hat{G},o)$ is unimodular. What about $(\hat{G}, o, \hat{\omega})$?

EXAMPLES OF RANDOM GRAPH MODELS

(1) $V = \{0,1\}^n$ $\quad v_i \sim v_j$ if $v_i - v_j = \pm e_k \quad 1 \le k \le n$
$\qquad e_k = (0 \cdots, \underset{k}{1}, \cdots 0)$

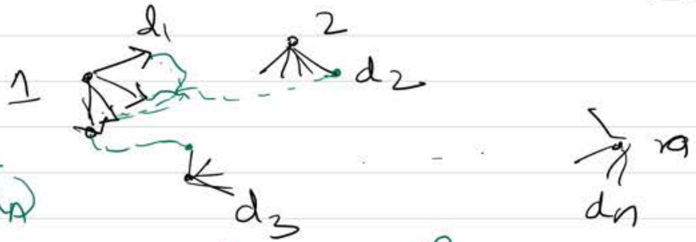
$p = c/n$. $H(n,p) =$ Each edge is preserved w.p. p & indep.

$\deg((0\cdots0)) \overset{d}{=} \mathrm{Bin}(n,p) \xrightarrow{d} \mathrm{Poi}(c)$.

$H(n,p) \xrightarrow{LW-P} BGW(\mathrm{Poi}(c))$??

(2) Configuration Model (vd Hofstad, Ch 3
Blaszczyszyn, Ch 4)

Let $\bar{d}_n = (d_{i,n})_{i=1}^{n}$ be a plausible degree sequence.
[Erdős–Gallai theorem]



$G(n, \bar{d}_n)$

Pair half-edges uniformly at random. Repeat until you get a simple graph.

$\deg(o) \overset{d}{=} \frac{1}{n}\sum_{i=1}^{n} \delta_{d_{i,n}}$

$\mathbb{P}(\deg(o) = k) = \underline{\#\{i : d_{i,n} = k\}} = p_{k,n}$.

Suppose $p_{k,n} \xrightarrow{n} p_k \ \forall k \geq 0$ & $\sum_{k=0}^{\infty} p_k = 1$.

Does $G(n, \bar{d}_n) \xrightarrow{LW-C} BGW(P)$? $P = (p_k)_{k\geq 0}$

But $BGW(P)$ is not unimodular unless $P \overset{d}{=} \mathrm{Poi}(\lambda)$.

But what is $P \neq \mathrm{Poi}(\lambda)$?

By unimod. considerations, $G(n, \bar{d}_n) \xrightarrow{LW-C} UGW(P)$.

1/12/2020 : L13 – Aldous-Steele Continuity Theorem

Let $\bar{G} = (V, E)$ be a finite connected graph, $\omega : V^2 \to \mathbb{R}_+$ an edge-symmetric weight function and $G = (\bar{G}, w)$ the associated weighted graph. We assume that for any subsets $A \neq B$ of E,

$$\sum_{e \in A} \omega(e) \neq \sum_{e \in B} \omega(e). \tag{6}$$

The *minimal spanning tree* $\mathrm{MST}(G)$ is the unique minimizer of

$$\tau(G) = \min_{T \in \mathrm{ST}(G)} \sum_{e \in E(T)} \omega(e), \tag{7}$$

where $\mathrm{ST}(G)$ is the set of spanning trees of $\bar{G}$ (uniqueness is a consequence of (6)). For $t > 0$, let $G(t) = (V, E(t))$, where $E(t) = \{e \in E : \omega(e) < t\}$ and for $v \in V$, let $G(t, v)$ be the connected component of v in $G(t)$. The following standard lemma gives a criterion to build to the minimal spanning tree.

Lemma 3.1. *An edge $e = \{u, v\} \in E$ belongs to $\mathrm{MST}(G)$ if and only if $G(\omega(e), u) \cap G(\omega(e), v) = \emptyset$.*

Lemma 3.1 can be used as a criterion to define the minimal spanning forest of an infinite graph.

Definition 3.2 (Minimal spanning forest). *Let $G = (\bar{G}, \omega)$ be a locally finite weighted graph with $\omega : V^2 \to \mathbb{R}_+$ edge-symmetric such that (6) holds for all finite subsets $A \neq B$ of edges in E. The minimal spanning forest of G, $\mathrm{MSF}(G)$ is the graph with vertex set V and edges, the set of $e \in E$ such that (i) $G(\omega(e), u) \cap G(\omega(e), v) = \emptyset$ and (ii) $G(\omega(e), u)$ and $G(\omega(e), v)$ are not both infinite.*

It is easy to check that $\mathrm{MSF}(G)$ is indeed a forest (no cycles). Also, by Lemma 3.1, if G is finite, our definition is consistent and $\mathrm{MSF}(G) = \mathrm{MST}(G)$. Finally, if G is an infinite graph, then each connected component of $\mathrm{MSF}(G)$ is infinite.

Aldous-Steele Continuity theorem :

Let G_n be a connected finite graph with edge-symmetric weights $\omega_n \in \mathbb{R}_+$ & satisfying (6) for all $n \geq 1$. Let $(G_n, \omega_n) \xrightarrow{\text{lw-d}} (G, \omega, o)$, a random rooted & weighted graph. Assume that (G, ω, o) satisfies (6) a.s.. Define (G_n, ω_n') by

$$\omega_n'(e) = (\omega_n(e), \mathbb{1}[e \in \mathrm{MST}(G_n)])$$

& (G, w') by $w'(e) = (w(e), \mathbb{1}[e \in MSF(G)])$.

Then $(G_n, w_n') \xrightarrow{LW-d} (G, w', o)$.

COROLLARY: Same assumptions as in the theorem.

Set $l_n(u) = \sum_{v \in V_n} w_n(u,v) \quad \forall \ u \in V_n$.

Assume that $l_n(o_n)$ is U.I. where $o_n \stackrel{d}{=} Unif(V_n)$.

$$\frac{\tau(G_n)}{|V_n|} \longrightarrow \frac{1}{2} \mathbb{E}\left[\sum_{v \in V} w(o,v) \mathbb{1}[(o,v) \in MSF(G)]\right]$$

So holds if $(G_n)_{n \geq 1}$ has bounded degrees & weights.

(i.e., $\frac{\tau(G)}{|V|}$ is estimable on bounded degree & weights graphs).

Proof: Given $(\bar{G}, \bar{o}, \bar{w}) \in \mathcal{G}_*(\underbrace{\mathbb{R}_+ \times \{0,1\}}_{=\Omega})$

$$L(\bar{G}, \bar{o}) = \frac{1}{2} \sum_{v \in V} \bar{w}(\bar{o}, v) \ \mathbb{1}[(o,v) \in MSF(\bar{G})] \leq \frac{l_n(o)}{2}$$

$$\mathbb{E} L(G_n, o_n) = \frac{\tau(G_n)}{|V_n|} = \frac{1}{|V_n|} \sum_{u \in V_n} L(G_n, u_n)$$

L is a continuous fn on $\mathcal{G}_*(\mathbb{R}_+ \times \{0,1\})$

As cty thm

$\Rightarrow \quad \mathbb{E}[\underbrace{L(G_n, o_n) \wedge t}_{\text{cts \& bdd}}] \longrightarrow \mathbb{E}[L(G, o) \wedge t] \quad \forall \ t > 0$. (1)

$$\mathbb{E}[L(G_n, o_n)] = \mathbb{E}[L(G_n, o_n) \wedge t] + \mathbb{E}[(L(G_n, o_n) - t) \ \mathbb{1}[L(G_n, o_n) \geq t]]$$

$$\mathbb{E}[(L(G_n, o_n) - t) \mathbb{1}[L(G_n, o_n) \geq t]] \leq \mathbb{E}[L(G_n, o_n) \mathbb{1}[\quad \geq t]]$$

$$\leq \frac{1}{2} \mathbb{E}[l_n(o_n) \mathbb{1}[l_n(o_n) \geq 2t]] \leq \varepsilon \quad \forall \ n \geq 1 \ \& \ t \text{ large.}$$

↳ U.I. of $l_n(o_n)$

Choose t large as above

$$\mathbb{E}[L(G_n,o_n)\wedge t] \le \mathbb{E}[L(G_n,o_n)] \le \mathbb{E}[L(G_n,o_n)\wedge t] + \varepsilon$$

$$① \Rightarrow \mathbb{E}[L(G,o)\wedge t] \le \lim_{n\to\infty} \frac{\tau(G_n)}{|V_n|} \le \mathbb{E}[L(G,o)\wedge t] + \varepsilon$$

By MCT, $\mathbb{E}[L(G,o)\wedge t] \uparrow \mathbb{E}[L(G,o)]$ as $t\uparrow\infty$.

$$\Rightarrow \mathbb{E}[L(G,o)] \le \lim_{n\to\infty} \frac{\tau(G_n)}{|V_n|} \le \mathbb{E}[L(G,o)] + \varepsilon. \qquad \blacksquare$$

THM: Let (G,o) be an a.s. ∞ unimodular random graph. Then $\mathbb{E}\deg_G(o) \ge 2$.

Proof: $e = (u,v) \in E$.

$\ell(u,v) = (f,f), (f,\infty), (\infty,f), (\infty,\infty)$

depending on whether component of u & v in $G\setminus\{e\}$ is finite or ∞.

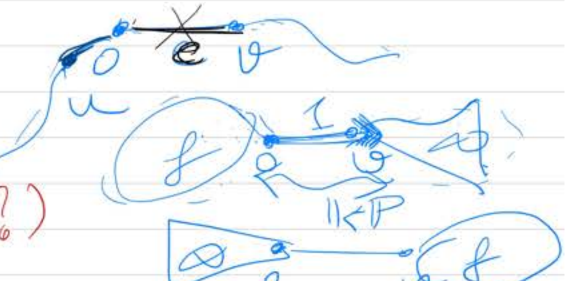
Let $a,b \in \{f,\infty\}$. D_{ab} = # of edges (o,v) ∋ $\ell(o,v) = (a,b)$.

a.s. $D_{ff} = 0$ as (G,o) is ∞ a.s.

If $\ell(o,v) = (\infty,\infty)$ then $\exists\, u$ ∋ $(o,u) \in E$ & $\ell(o,u) = (\infty,\infty)$

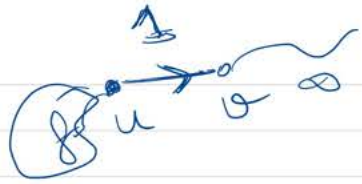
$\Rightarrow$ If $D_{\infty,\infty} \ge 1$ then $D_{\infty,\infty} \ge 2$.

If $D_{\infty,\infty} = 0$ then $D_{f,\infty} \ge 1$ (?)



$$\Rightarrow 2D_{f,\infty} + D_{\infty,\infty} \ge 2 \qquad — ①$$

$$f(G,u,v) = \mathbb{1}[\ell(u,v) = (f,\infty)]$$



By unimodularity,

$$\mathbb{E}\Big[\sum_{v\in V} f(G,0,v)\Big] = \mathbb{E}[D_{f,\infty}]$$

$$\| $$

$$\mathbb{E}\Big[\sum_{v\in V} f(G,v,0)\Big] = \mathbb{E}[D_{\infty,f}] \quad - \ ②$$

①+②

$$\Rightarrow \mathbb{E}[D_{\infty,f} + D_{f,\infty} + D_{\infty,\infty}] = \mathbb{E}[2D_{f,\infty} + D_{\infty,\infty}] \geq 2_0$$

$$(D_{ff} = 0 \text{ a.s.}) \leftarrow \mathbb{E}[\deg_G(0)]$$

$\blacksquare$

Proof of AS algthm:

$$(G_n, w_n) \xrightarrow{d} (G, w).$$

(G_n, w_n') converges along a subsequence

as $w_n' \in \mathbb{R}_+ \times \{0,1\}$ & so (G_n, w_n') is tight.

WLOG assume $(G_n, w_n') \xrightarrow{LW\text{-}d} (G, \tilde{w}, 0)$

where $\tilde{w}(e) = (w(e), \mathbb{1}[e \in S])$

for some $S \subseteq E(G)$.

[Instead of assuming $(G_{n_k}, w'_{n_k}) \xrightarrow{LW\text{-}d} (G, \tilde{w}, 0)$ we've assumed that the seq. converges for convenience]

If we show $\tilde{w} = w'$ ($\Longleftrightarrow S = MSF(G)$ a.s.)

then we get uniqueness of the limit &

thm follows.

STEP 1: $MSF(G) \subseteq S$.

we know $(G_n, w_n', 0_n) \xrightarrow{d} (G, \tilde{w}, 0)$

By Skorohod's representation thm, $\exists$ a prob. space

$\ni$ $d_*((G_n, w_n, o_n), (G, \tilde{w}, o)) \xrightarrow{n\to\infty} 0$ a.s. — (*1)

d_* - local weak metric on $\mathcal{G}_*(\Omega)$
$\Omega = \mathbb{R}_+ \times \{0,1\}$.

Let $e = (u,v) \in MSF(G) \Rightarrow G(w(e), u) \cap G(w(e), v) = \phi$
& one of them is finite.

WLOG assume $G(w(e), u)$ is finite.

$\exists$ t large $\ni$ $G(\underline{w(e)}, u) \subseteq B_t^G(o)$ [t is random dep. on (G, o)]

Because of uniqueness condn (6), we can choose $t \ni$

$G(w(e) + \frac{1}{t}, u) = G(w(e), u)$

From (*1), For any t $\exists$ n_0 (random) $\ni$

$\forall$ $n \geq n_0$, $B_{2t}^{G_n}(o_n) \overset{\phi}{\cong} B_{2t}^{G}(o)$

& $|w(e) - w(\phi(e))| \leq \frac{1}{2t}$; $\left(d_*((G_n, o_n, w_n), (G, o, w)) \leq \frac{1}{2t} \right)$

Let e_n be $\ni$ $\phi(e_n) = e$. $\Rightarrow |w(e_n) - w(e)| \leq \frac{1}{2t}$.
$e_n = (u_n, v_n)$ $\phi(u_n) = u$, $\phi(v_n) = v$.

$\Rightarrow \nexists$ no path from u to v in $G(w(e))$ i.e.,
$v \notin G(w(e), u) = G(w(e) + \frac{1}{t}, u) \subseteq B_t^G(o)$

Suppose $G(w(e_n), u_n) \not\subseteq B_t^{G_n}(o_n) \cong B_t^G(o)$

$\Rightarrow \exists$ $e_n' \in B_{t+1}(o_n) \setminus B_t(o_n)$ & $w(e_n') \leq w(e_n)$ & $e_n' \leftrightarrow u_n$.

So $w(\phi(e_n')) \leq w(e_n') + \frac{1}{2t} < w(e_n) + \frac{1}{2t} \leq w(e) + \frac{1}{t}$ — (*2)

$\phi(e_n') \leftrightarrow \phi(u_n) = u$

$\Rightarrow$ $\phi(e_n') \notin B_t^G(o)$ & $\phi(e_n') \in G(\underline{w(e)} + \frac{1}{t}, u)$
$\parallel$
a contradiction. $\Leftarrow$ $G(w(e), u) \subseteq B_t^G(o)$

$$\Rightarrow \underline{G(w(e_n), u_n)} \subseteq B_t^{G_n}(o_n)$$

$$\begin{array}{ccc} G(w(e_n), u_n) & \subseteq & B_t^{G_n}(o_n) \\ \| \int \phi & & \| \int \phi \\ H_n & \subseteq & B_t^{G}(o) \end{array}$$

Same argument as in (*2) $\Rightarrow H_n \subseteq G(w(e) + \frac{1}{t}, u)$.

$$\Rightarrow v \notin H_n \Rightarrow v_n = \phi^{-1}(v) \notin G(w(e_n), u_n)$$

$$\Rightarrow e_n = (u_n, v_n) \in MST(G_n)$$

i.e., $e_n \in MST(G_n) \quad \forall \; n \geq n_0$

& $MST(G_n) \longrightarrow S$ & $e_n \overset{\text{def}}{=} e$

$$\Longrightarrow e \in S.$$

Step 2 T.S.T. $MSF(G) \supseteq S$ a.s.

[If (G,o) is finite claim is easy. Assume (G,o) is ∞]

$$0 \leq \mathbb{E}\Big[\sum_{v \in V} \mathbb{1}[\{o,v\} \in S \setminus MSF(G)]\Big]$$

$$= \mathbb{E}\deg_S(o) - \mathbb{E}\deg_{MSF(G)}(o)$$

$$\leq 2 - 2 = 0 \quad \text{— } (*3)$$

[Fatou's lemma
$\mathbb{E}\deg_S(o) = \mathbb{E}\lim_{n\to\infty} \deg_{MST(G_n)}(o_n)$

$\leq \underline{\lim} \, \mathbb{E}\deg_{MST(G_n)}(o_n)$]

[$MSF(G)$ has all comp. ∞ & by prev lemma
$\mathbb{E}\deg_{MSF(G)}(o) \geq 2$]

$$\mathbb{E}\deg_{MST(G_n)}(o_n) = \frac{1}{|V_n|}\sum_{v \in V_n} \deg_{MST(G_n)}(v) = \frac{2\,|\text{Edges of } MST_{G_n}(o)|}{|V_n|} \leq \frac{2(|V_n|-1)}{|V_n|} \leq 2$$

(*3) $\Rightarrow \mathbb{P}(\exists\, v \sim o \ni (o,v) \in S \setminus MSF(G)) = 0.$

Proof to be concluded by the next lemma. ∎

LEMMA: [Everything shows at the root]

Let (G,o) be a unimodular randomly weighted rooted graph & $f: \mathcal{G}_* \to \{0,1\}$ be measurable.

If $f(G,o) = 1$ a.s., then a.s., $f(G,v) = 1 \; \forall v \in V$.

Proof: $h(G,u,v) = \mathbb{1}[f(G,u) \neq 1] : \mathcal{G}_{**} \to \{0,1\}.$

$$\mathbb{E}\Big[\sum_{v \in V} h(G,o,v)\Big] = \mathbb{E}\Big[\sum_{v \in V} \mathbb{1}[f(G,o) \neq 1]\Big] = 0$$

$\Vert$ MTP

$$\mathbb{E}\Big[\sum_{v \in V} h(G,v,o)\Big] = \mathbb{E}\Big[\sum_{v \in V} \mathbb{1}[f(G,v) \neq 1]\Big]$$

$$\Rightarrow \sum_{v \in V} \mathbb{1}[f(G,v) \neq 1] = 0 \quad \text{a.s.} \quad ∎$$

For prev. thm proof apply for

$$f(G,o) = \mathbb{1}[\nexists\, w \ni (o,w) \in S \setminus MSF(G)].$$

$f(G,v) = 1 \; \forall v \Rightarrow S \subseteq MSF(G).$

Sparse Random Graphs : Assignment 4

Yogeshwaran D.

December 6, 2020

Submit solutions to the below problems via Moodle by 26th December 10:00 PM.

1. Define the n-hypercube graph as follows : $V_n = \{0,1\}^n$ is the vertex set and edge set is $E_n = \{(v,w) : v - w = \overset{+}{-} e_k \text{for some } 1 \leq k \leq n\}$ i.e., (v,w) is an edge if they differ exactly at one co-ordinate. Let $H(n,p)$ denote the random graph such that each edge in E_n is chosen with probability $p \in [0,1]$ independently of each other. Let $O := (0,\ldots,0) \in V_n$ for all $n \geq 1$. For any $v \in V_k$, we set $v(n) := (v,0,\ldots,0) \in V_n$ for all $n \geq k$. Set $p_n = \min\{c/n, 1\}$ for all $n \geq 1$ where $c \in (0,\infty)$ and $H_n := H(n,p_n)$. What is the local weak limit of H_n ? First, describe the main steps in the proof and then give details for the individual steps.

2. Let (G,w,o) be a unimodular random weighted rooted graph with $w \in \mathbb{W}$, a Polish space. Let $A \subset \mathcal{G}_{**}(\mathbb{W})$ be an event invariant under re-rooting i.e., if $(G,w) \cong (G',w)$, then $(G,w,o) \in A$ iff $(G',w,o) \in A$. Assume that $\mathbb{P}(A) > 0$. Define a randomly weighted rooted graph (H,o) by the following probability distribution given by
$$\mathbb{P}((H,w',o) \in \cdot) = \mathbb{P}((G,w,o) \in .|(G,w,o) \in A).$$
Show that (H,w',o) is unimodular.

3. We define an end of a rooted tree (T,o) as a self-avoiding, semi-infinite path on T starting at o . Let (G,o) be a.s. an infinite graph. Show that if $\mathbb{E}[deg_G(o)] = 2$ thenshow that (G,o) is a.s. a tree and it has one or two ends a.s..

ADDITIONAL PROBLEMS (for practice) :

1. Show that sofic graphs are unimodular via Lusin's theorem.

2. Let $\mathbb{W}$, a Polish space be the mark space. Show that $\mathcal{G}_*(\mathbb{W}), \mathcal{G}_{**}(\mathbb{W})$ are complete seperable metric spaces.

3. Show that $(G,u,v) \mapsto (G,u)$ is a continuous map from $\mathcal{G}_{**}(\mathbb{W})$ to $\mathcal{G}_*(\mathbb{W})$.

4. Let (G,w,o) be a unimodular random weighted rooted graph with $w \in \mathbb{W}$, a Polish space. Suppose that $\phi : \mathcal{G}_{**}(\mathbb{W}) \to \mathbb{R}_+$ be a measurable function on the space of doubly rooted graphs. Set $w'(u,v) = \phi(G,u,v)$ for $(u,v) \in E(G)$. Show that (G,w',o) is a unimodular random weighted rooted graph.

5. Let (G, w, o) be a unimodular random weighted rooted graph with $w \in \mathbb{W}$, a Polish space. Let $B \subset \mathbb{W}$ be a Borel subset. Define a new weighted graph (G', w') where $V' = V$ and edge-set $E' = \{(u, v) : w(u, v) \in B, w(v, u) \in B\}$ and $w'(u, v) = w(u, v)$ for $(u, v) \in E'$. Show that (G', w', o) is a unimodular random weighted rooted graph.

6. Show that if G is an infinite graph, each connected component of $MSF(G)$ is infinite.

8/12 – L14 – MST COMPUTATIONS ON GW TREES

$GW(P)$ – $P = (P(k))_{k\geq 0}$ – pmf of the offspring random variable.

Assume $P(0) = P(1) = 0$.
$$EN^2 = \sum_{k\geq 2} k^2 P(k) < \infty.$$

$(T, o) \overset{d}{=} UGW(P)$

$P(\hat{N} = k-1) = \frac{kP(k)}{\mathbb{E}(N)}, \; k \geq 1$ $\quad N \quad \hat{N}$

$\omega = (\omega(e))_{e \in E(T)}$ i.i.d. $U[0,1]$.

$$\zeta_*(T) = \frac{1}{2}\mathbb{E}\sum_{v \in V} \omega(o,v)\,\mathbb{1}[(o,v) \in MSF(T)]$$

$$MSF(T) = MSF((T,\omega)).$$

[Assume $\exists\, G_n \ni (G_n, \omega) \xrightarrow{LWC} (T, o, \omega)$
& $\frac{T(G_n)}{|V_n|} \longrightarrow \zeta_*(T)$.]

$$2\zeta_* = \sum_{k\geq 1} kP(k)\, \mathbb{E}\big[\omega(o,1)\mathbb{1}[(o,1) \in MSF(T)] \,\big|\, N = k\big]$$
$$= \sum_{k\geq 1} kP(k) \int_0^1 t\, \mathbb{P}[\{o,1\} \in MSF(T) \,|\, \omega(o,1) = t, N = k]\,dt \quad \text{— (1)}$$

T_t^1 $\quad T \setminus (o,1) \quad \hat{T}_t$ $\quad \{\{o,1\} \notin MSF(T)\} = \{|T_t^1| = \infty, |\hat{T}_t| = \infty\}$

$T_t^1 = \{e \in T^1 : \omega(e) \leq t\}$ $\quad \hat{T}$ & T^1 are independent
$\hat{T}_t = \{e \in \hat{T} : \omega(e) \leq t\}$

$$(1) = \sum_{k\geq 1} kP(k) \int_0^1 t\big(1 - \mathbb{P}(|\hat{T}_t| = \infty)\, \mathbb{P}(|T_t^1| = \infty \,|\, N = k-1)\big)dt$$

$$= \sum_{k\geq 1} kP(k) \int_0^1 t\big(1 - \hat{q}(t)\, \mathbb{P}(|T_t^1| = \infty \,|\, N = k-1)\big)dt \quad \text{— (2)}$$

$\hat{q}(t) = 1 - \hat{p}(t)$, $\hat{p}(t) = p_{ext}(\hat{T}_t)$

MTP $\Rightarrow$
$$\sum_{k\geq 1} kP(k)\, \mathbb{P}(|T_t^1| = \infty \,|\, N = k-1) = EN \cdot \hat{q}(t)$$

$\varphi'(1) = EN = \sum_{k\geq 1} kP(k)$; $\varphi(x) = \mathbb{E}[x^N]$

$\hat{\varphi}(x) = \varphi'(x)/\varphi'(1)$ — $\hat{T}$

(1) + (2) $\Rightarrow$
$$2\zeta_* = \varphi'(1)\int_0^1 t\,(1 - \hat{q}(t)^2)\,dt$$

$\hat{p}(t)$, smallest root $r \ni \hat{\varphi}_t(r) = r$
$q(t) = 1 - \hat{\varphi}_t(1 - q(t))$ $\quad \hat{\varphi}_t = \mathbb{E}[x^{\hat{N}_t}]$
$= 1 - \hat{\varphi}(1 - t q(t))$

. . . . .
$$\Rightarrow \zeta_* = -\varphi'(1)\int_0^1 (1-x)\ln(1 - \hat{\varphi}(x))\,dx$$

[See lemma 3.7 of Bordenave "Counting unimodular ..." for full details]

1. $(T,o) = T_d$ – d-regular tree. $\varphi'(1) = d$, $\hat{\varphi}(x) = x^{d-1}$
$\Rightarrow$ $\zeta_*(T_d) = -d\int_0^1 (1-x)\ln(1 - x^{d-1})\,dx$.

2. $(T,o) \overset{d}{=} BGW(Poi(\lambda))$, $\hat{\varphi} = \varphi$, $\varphi'(1) = \lambda$
$$\zeta_*(\lambda) = -\lambda\int_0^1 (1-x)\ln(1 - e^{-\lambda(1-x)})\,dx$$
$$= -\int_0^1 (\lambda x)\ln(1 - e^{-\lambda x})\,dx$$

$$\zeta(1) = \int_0^1 t(1 - \hat{q}(t)^2)\,dt$$

$\hat{T} \overset{d}{=} T \overset{d}{=} BGW(Poi(1))$ $\quad T_t \overset{d}{=} \hat{T}_t \overset{d}{=} BGW(Poi(t))$ $\hookrightarrow$ by thinning property of Poisson r.v.

$\hat{q}(t) = q(t) = 1 - p_{ext}(BGW(t))$
$= 1 - e^{-t(1-q(t))}$

$$\frac{1}{2}\int_0^\infty t(1 - q(t)^2)\,dt = \zeta(3) \hookrightarrow \text{Riemann-Zeta fn.}$$

Eg: K_n – complete graph. $\omega(e)$ – i.i.d. $U[0,1]$
$\mathbb{E}[T(K_n)]$ $\quad \overset{d}{=} ER(n,t)$
$K_n(t) = (K_n, \{e : \omega(e) < t\}) = G(n,t)$, $t \in [0,1]$
$(u,v) \notin MST(K_n) \Leftrightarrow u \leftrightarrow v$ in $G(n, \omega(e))$
$$\mathbb{E}[T(K_n)] = \sum_{e \in K_n}\int_0^1 t\, P(u \not\leftrightarrow v \text{ in } G(n,t))\,dt$$
$(t = s/n)$ $\quad \approx \frac{1}{n^2}\sum_{e \in K_n}\int_0^{c\log n} s\, \mathbb{P}(u \not\leftrightarrow v \text{ in } G(n, s/n))\,ds$

[$\mathbb{P}(G(n,t)$ is connected$) \to 1$ if $t = \frac{\log n + b_n}{n}$, $b_n \to \infty$]

$$\approx \frac{1}{n^2}\sum_{e \in K_n}\int_0^\infty s\,\mathbb{P}(u \not\leftrightarrow v \text{ in } G(n,s/n))\,ds$$
$G(n, s/n)$ $\quad = \frac{1}{2}\int_0^\infty \mathbb{P}(u \not\leftrightarrow v \text{ in } G(n, s/n))\,ds$

$(G_n(s), u) \longrightarrow (T_s, u)$ $\quad T_s = GW(Poi(s))$
$(G_n(s), v) \longrightarrow (T_s', v)$ $\quad T_s' \overset{d}{=} T_s$
$\mathbb{P}(v \in B_r^{G_n(s)}(u)) \to 0 \quad \forall r \geq 1$
$\Rightarrow$ T_s & T_s' are indep & disjoint $GW(Poi(s))$ trees
$\Rightarrow$ $\{(u,v) \notin MSF(G(n, s/n))\}$
$\approx \{(u,v) \notin MSF(G_n(s))\}$
$= \{|T_s| = |T_s'| = \infty\}$
$\Rightarrow \mathbb{P}((u,v) \notin MSF(G(n,s/n))) \to \mathbb{P}(|T_s| = \infty)^2$
$= q(s)^2$

Frieze 1985. $\mathbb{E}[T(K_n)] \longrightarrow \frac{1}{2}\int_0^\infty s(1 - q(s)^2)\,ds = \zeta(3)$.

LWC Proofs (K_n, ω) $\omega(e)$ i.i.d. $EXP(1)$.
$\omega_n(e) = n\omega(e) \overset{d}{=} EXP(n)$.
$K_n' = (K_n, \omega_n)$ $\quad \frac{1}{n}T(K_n') = T(K_n)$

$K_n' \xrightarrow{LWC} PWIT(1)$
Poisson weighted ∞ tree.

S is a Poisson process on $\mathbb{R}_+$ with mean 1
i.e., $S_{i+1} - S_i$ are i.i.d. $EXP(1)$ r.v.'s.

$\mathbb{E}[\zeta_*(PWIT)] = \frac{1}{2}\mathbb{E}\big[\sum_{v \in V}\omega(o,v)\,\mathbb{1}[(o,v) \in MSF(PWIT)]\big]$

[See Pg 125 & 126 for more details] $\quad = \frac{1}{2}\int_0^\infty s\,\mathbb{P}((o,v) \in MSF(PWIT) \,|\, \omega(o,v) = s)\,ds$

$$= \frac{1}{2}\int_0^\infty s\,(1 - q(s)^2)\,ds = \zeta(3)$$

Objective method: Toy model for combinatorial optimization problems. Put i.i.d. $EXP(n)$ weights on K_n.
$\frac{\alpha_n}{n} = \frac{1}{n}\min_{H \subset K_n} \alpha(H) \longrightarrow \alpha(PWIT)$

$\rightarrow$ $G(n,p)$ – $|M|$ – size of maximal matching

$\frac{|M(G(n,p))|}{n} \longrightarrow ?$ ✓ LWC theory in Ch. 4 of Bordenave's notes.

Qn: $\frac{|IM(G(n,p))|}{n} \longrightarrow$??? $\quad$ IM – Induced matching

Matchings on Hypergraphs.

i.i.d. weights $U(0,1)$ $\quad \mathbb{E}[\text{min cost matching}] \to \zeta(2)$

. . . .

. . . . $\quad$ . . . $\quad \mathbb{E}[\quad] \to$???

SUMMARY.

- ER graph has many regimes
- It "looks like" a tree "locally" in the sparse regime
- Formalized via LWC - Extends to weighted graphs.

[Other Eg: Configuration Model, Hypercube percolation]
(See any of references) Assignment

- Limits are Unimodular RG.
- LWC $\Rightarrow$ bounded fnbs on bounded degree graphs are estimable!

$\Rightarrow$ Size of largest conn. component under some assumptions

$\Rightarrow$ Aldous-Steele Cty thm for MST. When limiting tree is GW, we can compute the limit.

See Bordenave - "Counting - - - " for more eg. of non-trivial estimable functions.

Is Spec(A_G) estimable?
Adj. matrix

LWC Link:

$$(K_n, \omega) \xrightarrow{LWC} ?$$

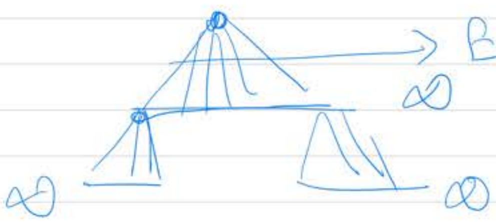
let $\omega(e)$ i.i.d. $EXP(1)$.

$$\mathbb{P}(\omega(e) \geq x) = e^{-x}, \quad x \geq 0$$

$$\mathbb{P}(\omega_n(e) \geq n^{-1}x) = e^{-x}, \quad \omega_n(e) = n\omega(e).$$

$\Rightarrow$ $\omega_n(e)$ i.i.d. $EXP(n)$

$$(K_n, \omega_n) \xrightarrow{LWC} PWIT(x)$$



$\to$ Edge weights $\stackrel{d}{=} \{S_1, S_2, \ldots\}$
$\quad = \mathcal{S}$

$\mathcal{S}$ = Poisson process on $\mathbb{R}_+$

i.e., $S_0 = 0$, $S_{i+1} - S_i$ are i.i.d. $EXP(1)$.

$$N_t = |\mathcal{S} \cap [0,t]| \stackrel{d}{=} Poi(t) \text{ r.v.}$$

$$\overset{d}{=} \{U_1, \ldots, U_{N_t}\}, \quad U_i \text{ i.i.d. } U[0,t]$$

$$\mathbb{E}[T_x(PWIT)] = \frac{1}{2}\mathbb{E}\sum_{v \in V} \omega(0,v)\, \mathbb{1}[(0,v) \in MSF(PWIT)]$$

$$= \frac{1}{2}\int_0^\infty s\, \mathbb{P}((0,v) \in MSF(PWIT) \mid \omega(0,v) = s)\, ds$$

$w(0,v) = s$

$(0,v) \in MST(PWIT) \iff |(PWIT(s),0)| < \infty$ or

$|(PWIT(s),v)| < \infty$

$\Rightarrow \mathbb{P}((0,v) \in MSF(PWIT)) = (1 - \mathbb{P}(|PWIT(s),0)| = \infty)^2)$

$(PWIT(s),0) \overset{d}{=} GW(Poi(s))$

since $|\partial n(0,s]| \overset{d}{=} Poi(s)$.

$\tau((K,w)) = \frac{1}{n}\tau((K_n,W_n))$

$\mathbb{E}[\tau((K,w))] = \frac{1}{n}\mathbb{E}[\tau((K_n,w_n))] \to \mathbb{E}[\tau_*(PWIT)]$

$= \zeta(3)$.