Proof of Corollary 5 :-

[Theorem 5]

Distributions of exchangeable random graphs $H \in \mathcal{L}_\infty$

$\updownarrow$ 1–1

Distributions of $\Gamma \in \mathcal{U}_\infty$.

Extreme points of Distributions of exchangeable random graphs $H \in \mathcal{L}_\infty$

$\updownarrow$ 1-1

Extreme points of

Distributions of $\Gamma \in \mathcal{U}_\infty$.

- Extreme points are delta measures on $\mathcal{U}_\infty$

- They are 1-1 corresponds with elements of $\mathcal{U}_\infty$

Take delta measure at point $\Gamma \in \mathcal{U}_\infty$

[Theorem 5] There is a $\mathbb{P}$ on $\mathcal{L}_\infty$ such that

$$\delta(F, P) = \mathbb{P}(H \supseteq F)$$

& so does the last conclusion. $\square$

Theorem 6 :- Let H be an exchangeable random infinite graph. Then TFAE

(i) The distribution of H is an extreme point in the set of exchangeable distributions in $\mathcal{L}_\infty$

(ii) If F_1 and F_2 are two finite graphs, $V(F_i) \subseteq \mathbb{N}$; $V(F_1) \cap V(F_2) = \phi$ then

$$\mathbb{P}(H \supseteq F_1 \cup F_2) = \mathbb{P}(H \supseteq F_1)\,\mathbb{P}(H \supseteq F_2)$$

(iii) $H|_{[k]}$ and $H|_{[k+1, \infty)}$ are independent for every k.

(iv) Let $\mathcal{F}_n$ be a σ-field generated by $H_{[n,\infty)}$

$\mathcal{F}_n \supseteq \mathcal{F}_{n+1}$

$\bigcap_{n=1}^{\infty} \mathcal{F}_n$ - tail σ-field is trivial, i.e. it contains elements of probability 0 or 1

Proof: (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) [done last class]

(iii) $\Rightarrow$ (iv) $\quad A \in \bigcap_{k=1}^{\infty} \mathcal{F}_k$

$\mathcal{G}_n := \sigma$-field generated by $H|_{[n]}$

$A \in \mathcal{F}_n \quad \forall n \geq 1$, $\quad \mathcal{F}_n$ - σ-field generated by $H|_{[n,\infty)}$

By (iii), A is independent of $\mathcal{G}_{n-1}$ $\forall n \geq 2$

$\Rightarrow$ A is independent of $\bigcup_{n=1}^{\infty} \mathcal{G}_n$
$= \mathcal{F}_1$

But $A \in \mathcal{F}_1$ $\Rightarrow$ A is independent of itself

$\Rightarrow$ $P(A \cap A) = P(A)^2$

$\Rightarrow$ $P(A)(1 - P(A)) = 0$

$\Rightarrow$ $P(A) \in \{0, 1\}$

$\Rightarrow$ $\bigcap_{k \geq 1} \mathcal{F}_k$ is trivial $\square$

(iv) $\Rightarrow$ (i) :- $\quad F \in \mathcal{L}_k$ for some k

$F_n \equiv F$ but vertex set is shifted by n

$$I = \mathbb{1}(H \supseteq F) \qquad I_n = \mathbb{1}(H \supseteq F_n)$$

$I_n \sim \mathcal{F}_n$ m'ble.

$$\mathbb{P}(H \supseteq F \cup F_n) = \mathbb{E}[I I_n]$$

$$= \mathbb{E}[\mathbb{E}[I I_n \mid \mathcal{F}_n]]$$

$$= \mathbb{E}[I_n \, \mathbb{E}[I \mid \mathcal{F}_n]]$$

Fact :- $\mathcal{F}_n \supseteq \mathcal{F}_{n+1}$ $\qquad Y_n = \mathbb{E}[I \mid \mathcal{F}_n]$

[Reverse or Backward Martingales]

$$\mathbb{E}[Y_n \mid \mathcal{F}_{n+1}] = \mathbb{E}[\mathbb{E}[I \mid \mathcal{F}_n] \mid \mathcal{F}_{n+1}]$$

$$= \mathbb{E}[I \mid \mathcal{F}_{n+1}] = Y_{n+1}$$

Convergence Theorem : $\quad Y_n \xrightarrow{a.s} \mathbb{E}[I \mid \bigcap_{n=1}^{\infty} \mathcal{F}_n]$

By D.C.T.

$$g_n = \mathbb{E}\left[\mathbb{E}[I \mid \mathcal{F}_n] - \mathbb{E}[I \mid \bigcap_{n=1}^{\infty} \mathcal{F}_n] \right] \longrightarrow 0$$

as $n \to \infty$

$$\mathbb{P}(H \supset F \cup F_n) = \mathbb{E}[I_n I] + g_n \quad \text{— } (*)$$

with $g_n \to 0$ as $n \to \infty$

$\underline{H\text{-exchangeable}}$ $\quad \mathbb{P}(H \supset F_n)$ does not depend on n $\equiv \mathbb{P}(H \supset F_k)$

$\underline{n \gg k}$

$\underline{\text{In } (*)}$ $\quad n \to \infty$

$$\mathbb{P}(F \supset F \cup F_k) = \mathbb{P}(H \supset F_k)\,\mathbb{P}(H \supset F)$$

$$= [\mathbb{P}(H \supset F)]^2 \quad \text{— } (F)$$

Γ be a random element of $\mathcal{U}_\infty$ $\overset{\text{(Theorem 5)}}{\longleftrightarrow}$ correspond to random exchangeable graph $H \in \mathcal{L}_\infty$

correspondence

$$\mathbb{E}(s(F, \Gamma)) = \mathbb{P}(H \supset F) \quad \text{— } (E)$$

(F) $\quad (\mathbb{E}\, s(F, \Gamma))^2 = (\mathbb{P}(H \supset F))^2 = \mathbb{P}(H \supset F \cup F_k)$

$$= \mathbb{E}\, s(F \cup F_k, \Gamma)$$

(E)

$$= \mathbb{E}\,[s(F, \Gamma)^2]$$

Earlier Corollary

$\mathbb{E}\,(s(F,\Gamma) \oplus s(F_k, \Gamma))$

$\mathbb{E}\,(s(F,\Gamma) \cdot s(F_k, \Gamma))$

$\mathbb{E}((s(F,\Gamma))^2)$

$\Rightarrow$ Variance of $s(F, \Gamma) = 0$

$\Rightarrow$ distribution of Γ is a delta measure

Corollary 5

$\Rightarrow$ H is an extreme point of exchangeable eulerian graphs

$\therefore$ (i) holds $\square$

So far: II^{nd} Half :- • Diaconis Janson – Section 6

• left is Section 7 [Uniqueness – Ex read]

0^{th} part :- – Dense graph, metric on this space $(\mathcal{F}, d_{sub})$

– [L S 06] – Compactifying limits of this ↗

$\boxed{1^{st}\ part}$:- LLN for dense graphs

$K : [0,1]^2 \to [0,1]$

$d_{sub}(G(n,K), K) \to 0$ as $n \to \infty$ as

$[\mathrm{II}^{nd}$ Half$]$:- Aldous – Hoover Theorem (assumed)

[Diaconis – Janson 08] paper till Section 6