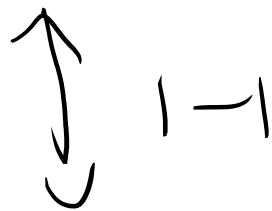


Proof of Corollary 5 :-

[Theorem 5]

Distributions of exchangeable
random graphs $H \in \mathcal{L}_\infty$

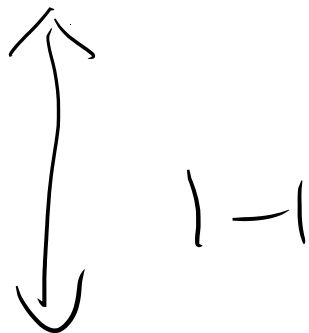


Distributions of $\pi \in \mathcal{U}_\infty$.

Extreme
points

Distributions of
exchangeable random graphs

$H \in \mathcal{L}_\infty$



Extreme points of

Distributions of $\mu \in \mathcal{U}_\mu$.

— Extreme points are delta measures on $\mathcal{U}_\mu$

— They are 1-1 corresponds with elements of $\mathcal{U}_\mu$

Take delta measure at point

$\mu \in \mathcal{U}_\mu$

[Theorem 5]

There is a $\mathbb{P}$ on $\mathcal{L}_\infty$ such that

$$s(F, \mathbb{P}) = \mathbb{P}(H \geq F)$$

& so does the last conclusion. $\square$

— Theorem 6 : — let H be an exchangeable random

infinite graph. Then TFAE

(i) The distribution of H is an extreme point in the set of exchangeable distributions in $\mathcal{L}_\infty$

(ii) If F_1 and F_2 are two finite graphs, $V(F_1) \subseteq \mathbb{N}$; $V(F_1) \cap V(F_2) = \emptyset$ then

$$\mathbb{P}(H \geq F_1 \cup F_2) = \mathbb{P}(H \geq F_1) \mathbb{P}(H \geq F_2)$$

(iii) $H|_{[k]}$ and $H|_{[k+1, \infty)}$ are independent for every k .

(iv) let $\mathcal{F}_n$ be a σ -field generated by $H|_{[n, \infty)}$

$$\mathcal{F}_n \supseteq \mathcal{F}_{n+1}$$

$\bigcap_{n=1}^{\infty} \mathcal{F}_n$ — tail σ -field is trivial, i.e. it contains elements of Probability 0 or 1

Proof:

(i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) [done last class]

$$(iii) \Rightarrow (iv) \quad A \in \bigcap_{k=1}^{\infty} \mathcal{F}_k$$

$\mathcal{G}_n =$ σ -field generated by $H|_{[n]}$

$A \in \mathcal{F}_n \quad \forall n \geq 1$, $\mathcal{F}_n$ - σ -field generated by $H|_{[n]}$

By (iii), A is independent of $\mathcal{G}_{n-1} \quad \forall n \geq 2$

$$\Rightarrow A \text{ is independent of } \bigcup_{n=1}^{\infty} \mathcal{G}_n = \mathcal{F}_1$$

But $A \in \mathcal{F}_1 \Rightarrow A$ is independent of itself

$$\Rightarrow P(A \cap A) = P(A)^2$$

$$\Rightarrow P(A)(1 - P(A)) = 0$$

$$\Rightarrow P(A) \in \{0, 1\}$$

$$\Rightarrow \bigcap_{k=1}^{\infty} \mathcal{F}_k \text{ is trivial } \square$$

(iv) $\Rightarrow$ (i) :- $F \in \mathcal{L}_K$ for some K .
 $F_n \equiv F$ but vertex set is shifted by n

$$I = \mathbb{1}(H \geq F) \quad I_n = \mathbb{1}(H \geq F_n)$$

$$I_n \sim I_n \text{ mible}$$

$$\begin{aligned} \mathbb{P}(H \geq F \cup F_n) &= E[I I_n] \\ &= E[E[I I_n | \mathcal{F}_n]] \\ &= E[I_n E[I | \mathcal{F}_n]] \end{aligned}$$

Fact :- $\mathcal{F}_n \supseteq \mathcal{F}_{n+1} \quad Y_n = E[I | \mathcal{F}_n]$

[Reverse or Backward Martingales]

$$\begin{aligned} E[Y_n | \mathcal{F}_{n+1}] &= E[E[I | \mathcal{F}_n] | \mathcal{F}_{n+1}] \\ &= E[I | \mathcal{F}_{n+1}] = Y_{n+1} \end{aligned}$$

Convergence Theorem : $Y_n \xrightarrow{a.s.} E[I | \bigcap_{n=1}^{\infty} \mathcal{F}_n]$

By D.C.T.

$$g_n = E[E[I | \mathcal{F}_n] - E[I | \bigcap_{n=1}^{\infty} \mathcal{F}_n]] \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\mathbb{P}(H \supset F \cup F_n) = \mathbb{E}[\mathbb{I}_H \mathbb{I}_F] + q_n \quad - \textcircled{4}$$

with $q_n \rightarrow 0$ as $n \rightarrow \infty$

H - exchangeable
 $n \gg k$

$\mathbb{P}(H \supset F_n)$ does not depend
on $n \equiv \mathbb{P}(H \supset F_k)$

in $\textcircled{4}$ $n \rightarrow \infty$

$$\mathbb{P}(F \supset F \cup F_k) = \mathbb{P}(H \supset F_k) \mathbb{P}(H \supset F)$$

$$= [\mathbb{P}(H \supset F)]^2 \quad - \textcircled{5}$$

P be a random element of $\mathcal{U}_n \xleftrightarrow{\text{Theorem 5}} \text{correspond to random exchangeable graph}$
 $H \in \mathcal{L}_n$

↙ correspondence

$$\mathbb{E}(\delta(F, P)) = \mathbb{P}(H \supset F) \quad - \textcircled{6}$$

$$\textcircled{7} \quad (\mathbb{E} \delta(P, \Gamma))^2 = (\mathbb{P}(H \supset F))^2 = \mathbb{P}(H \supset F \cup F_k)$$

$$= \mathbb{E} \delta(F \cup F_k, P)$$

$$\textcircled{8}$$

$$= \mathbb{E} (\delta(P, P)^2)$$

Cauchy Schwarz

$$\mathbb{E} (\delta(P, \Gamma) \delta(P, \Gamma_k))$$

$$\mathbb{E} (\delta(P, P) \delta(P, P))$$

$$\mathbb{E} (\delta(P, P)^2)$$

$$\Rightarrow \text{Variance of } \delta(P, P) = 0$$

$\Rightarrow$ distribution of Π is a delta measure

Corollary 5

$\Rightarrow$ H is an extreme point of exchangeable random graphs

$\therefore$ (i) holds $\square$

So far: IInd Half :- • Diaconis-Jarvis - Section 6

• left is Section 7 [Uniqueness - Ex read]

1st part :- Dense graph, metric on this space $(\mathcal{F}, d_{\text{sub}})$
- [Sob] - Compactifying limits of this $\uparrow$

1st part :- LLN for dense graphs
 $K: [0,1]^2 \rightarrow [0,1]$

$d_{\text{sub}}(G(n, K), K) \rightarrow 0$ as $n \rightarrow \infty$

(IInd Half) :- Aldous-Hoover Theory (assumed)
[Diaconis-Jarvis 08] paper till
Section 6