UC Berkeley, Spring 2021
CS 294-220

Posted: Jan 26
Due: Feb 10, 11PM PT

Problem Set 1

Solve all of the following problems. Before you start, make sure you are familiar with the course's Homework Policy.

1. Determine whether the following claim is true or false. Prove your answer.

Claim: Let $\mathcal{L}$ be a collection of languages, let $I = (L_i)_{i=1}^{\infty}$ be a computable indexing for $\mathcal{L}$, let M be a Turing machine that identifies $\mathcal{L}$ in the limit with respect to I, let $L \in \mathcal{L}$, and let $s = s_1, \dots, s_n$ be a sequence of strings that is a locking sequence for L that locks M onto $i \in \mathbb{N}$ such that $L_i = L$. Then $T = \{s_1, \dots, s_n\}$ is a telltale set for L with respect to $\mathcal{L}$. [7 pts]

2. Denote $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

(a) Let

$$\mathcal{L}_{\text{co-finite}} = \{\mathbb{N} \setminus X : \ X \subseteq \mathbb{N} \ \wedge \ |X| < \infty\},$$

and fix some computable indexing of $(L_i)_{i=1}^{\infty}$ of $\mathcal{L}_{\text{co-finite}}$. Consider the Turing machine M defined in Listing 1 below.[1]

Let $L = \mathbb{N} \setminus \{3\}$. Present a positive presentation p of L such that M diverges on p. [7 pts]

(b) Prove that $\mathcal{L}_{\text{co-finite}}$ is not identifiable in the limit.
Hint: Use Angluin's Theorem. [7 pts]

(c) For any $t \in \mathbb{N}$, let

$$\mathcal{L}_{t\text{-co-finite}} = \{\mathbb{N} \setminus X : \ X \subseteq \mathbb{N} \ \wedge \ |X| = t\}.$$

Prove that for any $t \in \mathbb{N}$, $\mathcal{L}_{t\text{-co-finite}}$ is identifiable in the limit.
Hint: Use Angluin's Theorem. [7 pts]

(d) For any $t \in \mathbb{N}$, define a Turing machine M_t that identifies $\mathcal{L}_{t\text{-co-finite}}$ in the limit, and prove that it does so correctly. [7 pts]

(e) Let $\mathcal{L} = \mathcal{L}_{1\text{-co-finite}} \cup \{\mathbb{N}\}$. Prove that $\mathcal{L}$ is not identifiable in the limit. [8 pts]

1 $M(s_1, s_2, \dots, s_n)$:
2 $\quad K \leftarrow \max\{s_1, s_2, \dots, s_n\}$
3 $\quad i \leftarrow$ `index` $i \in \mathbb{N}$ `such that` $L_i = \{s_1, s_2, \dots, s_n\} \cup \{k \in \mathbb{N} : \ k \geq K\}$
4 $\quad$ `output` i `and halt`

Listing 1 Turing machine M.

[1] M accepts as input sequences $s_1, s_2, \dots, s_n \in \mathbb{N}$ for any $n \in \mathbb{N}$. Assume that M is a well-defined Turing machine in the sense that the operation in line 3 (finding the appropriate index i with respect to the computable indexing) is computable.

3. Let $\mathcal{L} = \{\{1,2\}, \mathbb{N}\}$ and fix some computable indexing of $\mathcal{L}$. Consider the Turing machine A defined in Listing 2.

(a) Prove that A identifies $\mathcal{L}$ in the limit. [7 pts]

(b) Consider the following positive presentation of $\mathbb{N}$.

$$s = 0, 1, 2, 3, \ldots$$

Prove that no finite prefix $0, 1, 2, 3, \ldots, n$ of s is a locking sequence for $\mathbb{N}$ that locks A onto i such that $L_i = \mathbb{N}$. [8 pts]

(c) Explain why (a) and (b) do not contradict the Locking Sequence Lemma. [7 pts]

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A(s_1, s_2, ..., s_n):

    s_0 ← 0
    J ← {j ∈ {1,2,...,n}: s_{j-1} = 4 ∧ s_j = 3}

    if {s_1, s_2, ..., s_n} ⊆ {1,2} or J = {n}:
        i ← index i ∈ ℕ such that L_i = {1,2}
    else:
        i ← index i ∈ ℕ such that L_i = ℕ

    output i and halt
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Listing 2 Turing machine A.

4. Consider the following definition.

▶ **Definition 1.** *Let $\mathcal{L}$ be a set of languages over an alphabet Σ. We say that $\mathcal{L}$ has finite thickness if for every $s \in \Sigma^*$,*

$$|\{L \in \mathcal{L} : \; s \in L\}| < \infty.$$

In this question we will prove the following theorem.

▶ **Theorem 2** (Finite Thickness). *Let $\mathcal{L}$ be a set of languages over the same alphabet and let $(L_i)_{i=1}^{\infty}$ be computable indexing of $\mathcal{L}$. Assume that $\mathcal{L}$ has finite thickness. Then $\mathcal{L}$ is identifiable in the limit.*

Let $s = s_1, s_2, \ldots$ be a computable enumeration of Σ^* (that is, s is a positive presentation of Σ^* such that there exists a Turing machine that never halts and that prints out all the strings in s in order one by one). For any positive integers n and language $L \in \mathcal{L}$, let $L^{(n)} = \{s_1, s_2, \ldots, s_n\} \cap L$.

Fix a set $\mathcal{L}$ that satisfies the assumptions of the theorem. Let i be a positive integer. Let $t \in L_i$, and let T be a finite set such that $t \in T \subseteq L_i$.

(a) Let $B(T) = \{L \in \mathcal{L} : \; T \subseteq L \subsetneq L_i\}$. Show that $|B(T)| < \infty$. [7 pts]

(b) Show that if $L \in B(T)$, then there exists $n \in \mathbb{N}$ such that $T \subseteq L^{(n)} \subsetneq L_i^{(n)}$. [7 pts]

(c) Let j be a positive integer. Present an algorithm that takes i, j and T as input and operates as follows. If there exists a positive integer n such that $T \subseteq L_j^{(n)} \subsetneq L_i^{(n)}$, then the algorithm halts and outputs a string $x \in L_i$ such that $x \notin L_j$. Otherwise, the algorithm never halts. [7 pts]

(d) Present an algorithm that takes i as input, and prints out a finite or infinite sequence of strings $t_1, t_2, t_3, \ldots$ such that the set $T = \{t_1, t_2, t_3, \ldots\}$ is finite and is a telltale set for L_i with respect to $\mathcal{L}$. [7 pts]

(e) Prove Theorem 2 using the previous steps. [7 pts]