Random Walks on Graphs / Topics in Applied Stochastic Process : Assignment 1

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Submit solutions to Q1, Q6, Q7, Q8 on Moodle by 17th February 10:00 PM.

1. Let P, Q be $V \times V$ stochastic matrices for a finite set V. Let PQ be the usual matrix multiplication i.e.,

$$PQ(x, y) = \sum_{z \in V} P(x, z)Q(z, y).$$

   (a) Show that PQ is also a stochastic matrix.

   (b) If P, Q are reversible with respect to a measure μ and $PQ = QP$, then PQ is also reversible with respect to μ.

2. Compute the eigenvalues of the Laplacian for the path graph P_n, cycle graph C_n, complete graph K_n and the complete bi-partite graph $K_{m,n}$.

3. Show that $\lambda_{n-1} = 2$ iff G is bi-partite.

4. Consider the hypercube graph on $V = \{-1, +1\}^n$ with the following weight function : $\mu(x, y) = \frac{1[x \sim y]}{n 2^{n+1}}, \mu(x, x) = \frac{1}{2^{n+1}}$. What is the Poincaré inequality for this graph ?

5. Let $G = K_k^n$ i.e., the product of n complete graphs. What is the Poincaré inequality for this graph ?

6. Consider the Dirichlet problem on a finite, connected and weighted graph (G, μ). Let $\Omega = V$. Show that for a given function $f : V \to \mathbb{R}$, if a solution u exists for the Dirichlet problem, then infinitely many solutions exist. Further, a solution exists for a given function $f : V \to \mathbb{R}$ iff $\sum_{x \in V} f(x)\mu(x) = 0$.

7. Let $P_i, i = 1, 2$ be stochastic matrices on V_i. Consider the following matrix $P = P_1 \otimes P_2$ on $V = V_1 \times V_2$ defined as

$$P((x, y), (z, w)) = P_1(x, z)P_2(y, w).$$

   Find the eigenvalues and eigenfunctions of P in terms of the eigenvalues and eigenfunctions of P_1 and P_2.

8. Let $X_1, \ldots, X_n, X_1', \ldots, X_n'$ be i.i.d. $\{-1, +1\}$ random variables such that $\mathbb{P}(X_1 = 1) = p = 1 - \mathbb{P}(X_1 = -1)$ where $p \in (0, 1)$. Show that for $f : \{-1, +1\}^n \to \mathbb{R}$, we have that

$$\mathbb{VAR}[f(X_1, \ldots, X_n)] \leq \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}[(f(X_1, \ldots, X_i, \ldots, X_n) - f(X_1, \ldots, X_i', \ldots, X_n))^2].$$

HINT : Construct a suitable reversible stochastic matrix on $V = \{-1, +1\}^n$.

9. Generalize the above problem to the case of X_i's taking values in a finite set V with a pmf π such that $\pi(x) > 0$ for all $x \in V$.